J. reine angew. Math., Ahead of Print
DOI 10.1515/crelle-2013-0107

Journal für die reine und angewandte Mathematik

Mahler measure and elliptic curve L-functions at $s = 3$

By *Matilde Lalín* at Montreal

Abstract. We study the Mahler measure of some three-variable polynomials that are conjectured to be related to L-functions of elliptic curves at $s = 3$ by Boyd. The connection with L-functions can be explained with the use of a regulator and a result of Goncharov. Finally, we prove a relationship between two formulas.

1. Introduction

Let $P(x_1, \dots, x_n) \in \mathbb{C}[x_1^{\pm}, \dots, x_n^{\pm}]$ be a nonzero Laurent polynomial. The (logarithmic) Mahler measure of $P(x_1, \dots, x_n)$ is given by

$$\mathrm{m}(P(x_1, \dots, x_n)) = \int_0^1 \cdots \int_0^1 \log |P(e^{2\pi i \theta_1}, \dots, e^{2\pi i \theta_n})| d\theta_1 \dots d\theta_n.$$

The Mahler measure of the polynomial family $P_k(x, y) = x + \frac{1}{x} + y + \frac{1}{y} + k$ (where k is a parameter) was studied by Boyd [4] who found numerically that

$$\mathrm{m}\left(x + \frac{1}{x} + y + \frac{1}{y} + k\right) \stackrel{?}{=} \frac{L'(E_{N(k)}, 0)}{s_k}, \tag{1.1}$$

where k is an integer different from 0 and 4, s_k is a rational number (often integer), and $E_{N(k)}$ is the elliptic curve (of conductor $N(k)$) resulting from the zero set of the polynomial P_k. From now on, the question mark stands for equalities that have been numerically verified up to at least 28 decimal places.

The connection between the Mahler measure of P_k and $L'(E_{N(k)}, 0)$ was predicted by Deninger [9] who explained it in terms of Beĭlinson's conjectures [1].

In [18], Rodriguez-Villegas expressed this Mahler measure as an Eisenstein–Kronecker series:

$$\begin{aligned}\mathrm{m}\left(x + \frac{1}{x} + y + \frac{1}{y} + k\right) &= \mathrm{Re}\left(\frac{16 \,\mathrm{Im}\, \tau}{\pi^2} \sum_{m,n}{}' \frac{\chi_{-4}(m)}{(m + n4\tau)^2 (m + n4\bar{\tau})}\right) \\ &= \mathrm{Re}\left(-\pi i \tau + 2 \sum_{n=1}^{\infty} \sum_{d \mid n} \chi_{-4}(d) d^2 \frac{q^n}{n}\right),\end{aligned}$$

where the parameters τ and q are coming from

$$q = e^{2\pi i\tau} = q\left(\frac{16}{k^2}\right) = \exp\left(-\pi \frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1,1-\frac{16}{k^2}\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1,\frac{16}{k^2}\right)}\right).$$

This formula can be connected to the elliptic dilogarithm (using ideas of Bloch [3]) which in favorable cases relates to the special value of the L-function via Beĭlinson's conjectures.

The question mark may be removed from equation (1.1) in instances where Beĭlinson's conjectures are known. For example, Rodriguez-Villegas [18] proved

$$\mathrm{m}\left(x + \frac{1}{x} + y + \frac{1}{y} + 4\sqrt{2}\right) = L'(E_{32}, 0). \tag{1.2}$$

It suffices that k^2 be an integer for equation (1.1) to have an interpretation in terms of Beĭlinson's conjectures. In this particular case, the curve has complex multiplication.

Other examples were given by Rogers and Zudilin: in [20] they proved

$$\mathrm{m}\left(x + \frac{1}{x} + y + \frac{1}{y} + 1\right) = \frac{15}{4\pi^2}L(E_{15}, 2) = L'(E_{15}, 0), \tag{1.3}$$

and in [21] they proved

$$\mathrm{m}\left(x + \frac{1}{x} + y + \frac{1}{y} + 8\right) = \frac{24}{\pi^2}L(E_{24}, 2) = 4L'(E_{24}, 0). \tag{1.4}$$

These two examples also correspond to known cases of Beĭlinson's conjectures (for modular curves). Other identities of the same type in different polynomial families (also originated from Boyd's work [4]) were proved by Brunault [6, 7] and Mellit [17].

Before such equalities were proved, there was a surge of results (started by Rodriguez-Villegas [19] and then continued by Bertin [2] and others) relating Mahler measures of different polynomials in the same family via functional equations. From these equations, identities between Mahler measures (originally conjectured by Boyd in [4]) were finally proved, such as

$$\mathrm{m}\left(x + \frac{1}{x} + y + \frac{1}{y} + 8\right) = 4\mathrm{m}\left(x + \frac{1}{x} + y + \frac{1}{y} + 2\right), \tag{1.5}$$

proved in [14], and

$$\mathrm{m}\left(x + \frac{1}{x} + y + \frac{1}{y} + 5\right) = 6\mathrm{m}\left(x + \frac{1}{x} + y + \frac{1}{y} + 1\right), \tag{1.6}$$

proved in [13].

In this note, we work with the following numerical identities also found by Boyd [5]:

$$\mathrm{m}(z + (x+1)(y+1)) \stackrel{?}{=} 2L'(E_{15}, -1), \tag{1.7}$$

$$\begin{aligned} \mathrm{m}((x+1)z + (x^2+x+1)y + (x+1)^2) &= m_1 + m_2 \\ &\stackrel{?}{=} \frac{1}{3}L'(\chi_{-3}, -1) + \frac{13\zeta(3)}{3\pi^2}, \end{aligned} \tag{1.8}$$

and the exotic numerical relation

$$m_1 - m_2 \stackrel{?}{=} L'(\chi_{-3}, -1) - L'(E_{15}, -1). \tag{1.9}$$

The terms m_1 and m_2 come from the way the integration is performed and will be made precise in Section 3. The relationship between these polynomials and the elliptic curves of conductor 15 will be clarified in Section 2.

One of the goals of these notes is to prove a result in the spirit of equations (1.5)–(1.6):

Theorem 1. *We have*

$$\mathrm{m}(z + (x+1)(y+1)) = 2L'(\chi_{-3}, -1) + 2m_2 - 2m_1.$$

We stress that the result of Theorem 1 is exact, not numerical, and it corresponds to an identity relating two formulas (in the style of equations (1.5) and (1.6)) where each side is expected to involve $L'(E,-1)$ rather than $L'(E,0)$.

Our techniques involve relating the Mahler measure to a regulator and comparing the value of the regulator in two different elements of $K_4(E)$ for E an elliptic curve. The relationship with $L'(E,-1)$ can be explained by using a result conjectured by Deninger [8] and proved by Goncharov [11].

For a field F, let

$$B_2(F) = \mathbb{Z}[F^*]/R_2(F)$$

be the Bloch group, where $R_2(F)$ is the subgroup of $\mathbb{Z}[F^*]$ generated by the five-term relation

$$[x] + [y] + [1 - xy] + \left[\frac{1-x}{1-xy}\right] + \left[\frac{1-y}{1-xy}\right].$$

We also set $[0] = [\infty] = 0$ when we need to work in $\mathbb{P}^1(F)$.

Remark that the five-term relation is expected to generate the rational functional equations of the Bloch–Wigner dilogarithm, defined later in (2.1) (see [10] for more details).

We are now ready to state our main result.

Theorem 2. *Let $P \in \mathbb{Q}[x,y,z]$ be nonreciprocal and let*

$$\{\mathrm{Res}_z(P(x,y,z), P(x^{-1},y^{-1},z^{-1})) = 0\}$$

correspond to an elliptic curve E. Assume that $K_4(E)$ has rank 1 and that

$$\{|x| = |y| = |z| = 1\}$$

corresponds to a real cycle in E. Suppose further that there exist x_j, y_j such that

$$x \wedge y \wedge z = \sum_j x_j \wedge (1 - x_j) \wedge y_j = 0 \quad \text{in} \quad \bigwedge^3 \mathbb{Q}(E)^*$$

and that

$$\sum_j v_z(y_j)[x_j(z)] = 0 \quad \text{in } B_2(\overline{\mathbb{Q}}) \quad \text{for all } z \in E(\overline{\mathbb{Q}}).$$

Then there is a rational number q such that

$$\mathrm{m}(P) = \mathrm{m}(P^*) + qL'(E,-1),$$

where P^* is a two-variable polynomial.

Beĭlinson's conjectures predict that the dimension of $K_4(E)$ will be equal to the order of vanishing of $L(E,s)$ at $s = -1$, which is 1 in our cases.

We will see in Section 4.1 that Theorem 2 implies that equation (1.7) is true up to a rational factor. In other words, we can prove that

$$\mathrm{m}(z + (x+1)(y+1)) = qL'(E_{15}, -1),$$

where q is a rational number, conditionally on the conjecture that, for the elliptic curve

$$E : Y^2 = X^3 - 7X^2 + 16X,$$

the dimension of $K_4(E)$ is equal to 1. In addition, we conjecture that the rational factor q equals 2.

2. Mahler measure and the regulator

2.1. The two-variable case. We briefly discuss the situation for the Mahler measure of a two-variable polynomial P and its relation to $L'(E,0)$. See [9, 18] for details.

By Jensen's formula, one can write, for any such P,

$$\mathrm{m}(P) = \mathrm{m}(P^*) - \frac{1}{2\pi}\int_{\gamma_0} \eta(x,y),$$

where $\gamma_0 = \{P(x,y) = 0\} \cap \{|x| = 1, |y| \geq 1\}$,

$$\eta(x,y) := \log|x| d \arg y - \log|y| d \arg x$$

is a multiplicative and antisymmetric closed form and P^* is a one-variable polynomial. More precisely, if we write $P(x,y) = a_n(x)y^n + \cdots + a_0(x) \in \mathbb{C}[x][y]$, then $P^*(x) = a_n(x)$. In our applications we will have that $\mathrm{m}(P^*) = 0$ and this term will be ignored.

Let $E/\mathbb{C}$ be an elliptic curve. For example, it can be the curve determined by the condition $P_k(x,y) = 0$. By Matsumoto's theorem the K_2-group can be built as

$$K_2(\mathbb{C}(E)) \cong \Lambda^2 \mathbb{C}(E)^\times / \{x \otimes (1-x)\}.$$

Under certain conditions that will be verified in our cases (the triviality of tame symbols, see [18]), we can think of $K_2(E) \otimes \mathbb{Q} \subset K_2(\mathbb{C}(E)) \otimes \mathbb{Q}$.

The regulator map is defined by

$$r : K_2(E) \otimes \mathbb{Q} \to H^1(E, \mathbb{R}), \quad \{x,y\} \to \left\{\gamma \to \int_\gamma \eta(x,y)\right\}$$

for $\gamma \in H_1(E, \mathbb{Z})$.

Here we think of $H^1(E,\mathbb{R})$ as the dual of $H_1(E,\mathbb{Z})$. The function is well defined because $\eta(x, 1-x) = dD(x)$ where

$$D(z) = \mathrm{Im}(\mathrm{Li}_2(z)) + \arg(1-z)\log|z| \tag{2.1}$$

is the Bloch–Wigner dilogarithm.

We can summarize the above discussion by writing

$$\mathrm{m}(P) = \mathrm{m}(P^*) - \frac{1}{2\pi} r(\{x, y\})[\gamma_0]. \tag{2.2}$$

It remains to see how to compute $r(\{x, y\})[\gamma_0]$.

Let $E/\mathbb{C} = \mathbb{C}/\Lambda$ an elliptic curve with complex period lattice $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ (where $\mathrm{Im}(\frac{\omega_2}{\omega_1}) > 0$). Let $A(\Lambda) = \frac{\overline{\omega_1}\omega_2 - \omega_1\overline{\omega_2}}{2\pi i}$ (thus $2\pi i A(\Lambda)$ is the volume of Λ). The Pontrjagin pairing is defined between the elliptic curve and the lattice Λ:

$$(\cdot,\cdot) : \mathbb{C}/\Lambda \times \Lambda \to \mathbb{S}^1, \quad (P, \gamma) := \exp\left(\frac{P\overline{\gamma} - \overline{P}\gamma}{A(\Lambda)}\right).$$

In other words, if $P = a\omega_1 + b\omega_2$ and $\gamma = m\omega_1 + n\omega_2$,

$$(P, \gamma) = e^{2\pi i(bm - an)}.$$

This pairing can be extended to divisors in the obvious way. If $x = \sum r_j(P_j)$, then

$$(x, \gamma) := \prod (P_j, \gamma)^{r_j}.$$

Following Bloch [3] and Beĭlinson [1], define the regulator function by an Eisenstein–Kronecker series

$$R_E(P) = A(\Lambda)^2 \sum_{\gamma}{}' \frac{(P, \gamma)\overline{\gamma}}{|\gamma|^4}. \tag{2.3}$$

Then we have the following result.

Theorem 3 (Beĭlinson [1]). *Let $E/\mathbb{C}$ be an elliptic curves, $x, y \in \mathbb{C}(E)^*$, $\omega \in \Omega^1(E)$. Then*

$$\int_{E(\mathbb{C})} \log|y| d \arg x \wedge \overline{\omega} = \pi R_E((x) \diamond (y)),$$

where the diamond operation $\diamond : \mathbb{C}(E)^* \otimes \mathbb{C}(E)^* \to \mathbb{Z}[E(\mathbb{C})]^-$ is defined on the divisors (x) and (y) as

$$(x) \diamond (y) = \sum r_i s_j (P_i - Q_j).$$

for

$$(x) = \sum r_i(P_i), \quad (y) = \sum s_j(Q_j).$$

From this, one can deduce a result at the level of the Mahler measure.

Corollary 4 ([14, Corollary 3.2]). *Let E be the elliptic curve defined by $P = 0$. If x and y are non-constant functions in $\mathbb{C}(E)$ with trivial tame symbols, then*

$$\mathrm{m}(P) - \mathrm{m}(P^*) = \frac{1}{A(\Lambda)} \mathrm{Im}(\Omega R_E((x) \diamond (y))), \tag{2.4}$$

where $\Omega = \int_{\gamma_0} \omega$ and P^* is a one-variable polynomial.

Thus, relationships among Mahler measures for polynomials in certain families defining elliptic curves may be deduced from relationships for the divisors of x and y. Moreover, in favorable cases, the right-hand side of equation (2.4) can be directly related to $L(E, 2)$, thus leading to formulas such as (1.2), (1.3), and (1.4). In that case the above discussion may be summarized as

$$\mathrm{m}(P) - \mathrm{m}(P^*) \sim_{\mathbb{Q}} \frac{\operatorname{Im}(\Omega)}{\Omega_{\mathbb{R}}} L'(E,0), \tag{2.5}$$

where the symbol $\sim_{\mathbb{Q}}$ means up to an unknown rational factor and $\Omega_{\mathbb{R}} = \int_{E(\mathbb{R})} \omega$ is the real period of E.

2.2. The three-variable case and the proof of Theorem 2. Now suppose we have a three-variable polynomial P. By application of Jensen's formula,

$$\mathrm{m}(P) = \mathrm{m}(P^*) - \frac{1}{(2\pi)^2} \int_{\Gamma_0} \eta(x,y,z),$$

where $\Gamma_0 = \{P(x,y,z) = 0\} \cap \{|x| = |y| = 1, |z| \geq 1\}$,

$$\begin{aligned} \eta(x,y,z) = \log|x|\big(&d \log|y| \wedge d \log|z| - d \arg y \wedge d \arg z\big) \\ &+ \log|y|\big(d \log|z| \wedge d \log|x| - d \arg z \wedge d \arg x\big) \\ &+ \log|z|\big(d \log|x| \wedge d \log|y| - d \arg x \wedge d \arg y\big), \end{aligned}$$

and P^* is a two-variable polynomial. More precisely, if we write the polynomial P in the form $P(x,y,z) = a_n(x,y)z^n + \cdots + a_0(x,y) \in \mathbb{C}[x,y][z]$, then

$$P^*(x,y) = a_n(x,y).$$

In our cases we will have that $\mathrm{m}(P^*) = 0$ and this term will be ignored. (See [12] for more details.)

As in the two-variable case, this differential form is also multiplicative, antisymmetric, closed, and it satisfies

$$\eta(x, 1-x, y) = d\rho(x,y),$$

where

$$\rho(x,y) = -D(x) d \arg y + \frac{1}{3} \log|y| \big(\log|1-x| d \log|x| - \log|x| d \log|1-x|\big).$$

Suppose that the following condition is satisfied:

$$x \wedge y \wedge z = \sum_j x_j \wedge (1-x_j) \wedge y_j \tag{2.6}$$

in $\bigwedge^3(\mathbb{C}(S)^*) \otimes \mathbb{Q}$ where $S = \{P(x,y,z) = 0\}$. In this case η is exact and we apply Stokes' theorem to further compute the integral. Thus, we obtain

$$\int_{\Gamma_0} \eta(x,y,z) = \sum \int_{\Gamma_0} \eta(x_j, 1-x_j, y_j) = \sum_j \int_{\partial\Gamma_0} \rho(x_j, y_j),$$

where

$$\Gamma_0 = \{P(x,y,z) = 0\} \cap \{|x| = |y| = 1, |z| \geq 1\}.$$

The set $\partial\Gamma_0$ may be obtained by setting $|z| = 1$ as described above and it may contain singularities. Our goal is to desingularize and algebraically describe this set. We thus change our point of view. Namely, assume that $P \in \mathbb{Q}[x, y, z]$ and is nonreciprocal (this condition is true for the examples we study); then, on the unit torus $|x| = |y| = |z| = 1$, we can write

$$\partial\Gamma_0 = \{P(x, y, z) = P(x^{-1}, y^{-1}, z^{-1}) = 0\} \cap \{|x| = |y| = |z| = 1\}.$$

(This idea was proposed by Maillot [16].) Observe that we are integrating now on a path $\{|x| = |y| = 1\}$ inside the curve

$$C = \{\mathrm{Res}_z(P(x, y, z), P(x^{-1}, y^{-1}, z^{-1})) = 0\}.$$

While η is naturally evaluated on $\bigwedge^3 \mathbb{Q}(S)^*$, ρ is naturally evaluated on $B_2(\mathbb{Q}(C)) \otimes \mathbb{Q}(C)^*$, because the first argument x behaves like the argument of the dilogarithm and the second argument y behaves multiplicatively.

We are interested in the case where C is an elliptic curve E. We will use the following result due to Goncharov.

Theorem 5 ([11, Theorem 3.4]). *Let $E/\mathbb{C}$ be an elliptic curve, $x_j, y_j \in \mathbb{C}(E)^*$ satisfy the condition*

$$\sum_j x_j \wedge (1 - x_j) \wedge y_j = 0 \quad \textit{in } \bigwedge^3 \mathbb{C}(E)^*.$$

Then

$$\sum \int_{E(\mathbb{C})} \log|y_j| \big(\log|1 - x_j| d\log|x_j| - \log|x_j| d\log|1 - x_j|\big) \wedge \omega$$
$$= \sum_j \sideset{}{'}\sum_{\gamma_1+\gamma_2+\gamma_3=0} \frac{(P_j, \gamma_1)(Q_j, \gamma_2)(R_j, \gamma_3)(\overline{\gamma_3} - \overline{\gamma_2})}{|\gamma_1|^2|\gamma_2|^2|\gamma_3|^2},$$

where $\sum'$ indicates that we exclude the term with all $\gamma_i = 0$ from the sum, $\omega \in \Omega^1(E)$ is normalized by $\int_{E(\mathbb{C})} \omega \wedge \overline{\omega} = 1$ and P_j, Q_j, R_j are the divisors of $y_j, x_j, 1 - x_j$ respectively.

Corollary 6. *Let $P \in \mathbb{Q}[x, y, z]$ be nonreciprocal such that*

$$\{\mathrm{Res}_z(P(x, y, z), P(x^{-1}, y^{-1}, z^{-1})) = 0\}$$

corresponds to an elliptic curve E. Suppose in addition that

$$x \wedge y \wedge z = \sum_j x_j \wedge (1 - x_j) \wedge y_j = 0 \quad \textit{in } \bigwedge^3 \mathbb{Q}(E)^*,$$

and

$$\sum_j v_z(y_j)[x_j(z)] = 0 \quad \textit{in } B_2(\overline{\mathbb{Q}}) \quad \textit{for all } z \in E(\overline{\mathbb{Q}}).$$

Then

$$\mathrm{m}(P) - \mathrm{m}(P^*) = \frac{2}{3\pi^2} \mathrm{Re}(\Omega) \sum_j \sideset{}{'}\sum_{\gamma_1+\gamma_2+\gamma_3=0} \frac{(P_j, \gamma_1)(Q_j, \gamma_2)(R_j, \gamma_3)(\overline{\gamma_3} - \overline{\gamma_2})}{|\gamma_1|^2|\gamma_2|^2|\gamma_3|^2},$$

where $\Omega = \int_{\partial\Gamma_0} \omega$ and P_j, Q_j, R_j are the divisors of $y_j, x_j, 1 - x_j$ respectively.

Proof. First note that Goncharov proves

$$\int_{E(\mathbb{C})} \log|y|\big(\log|1-x|d\log|x| - \log|x|d\log|1-x|\big)\wedge\omega = -\int_{E(\mathbb{C})} D(x)d\arg y\wedge\omega.$$

From this and the definition of $\rho(x,y)$ one gets

$$\int_{E(\mathbb{C})} \rho(x,y)\wedge\omega = \frac{4}{3}\int_{E(\mathbb{C})} \log|y|\big(\log|1-x|d\log|x| - \log|x|d\log|1-x|\big)\wedge\omega.$$

Now we may observe that $\sum_j \rho(x_j,y_j)$ is an element of the one-dimensional vector space $H^2_{\mathcal{D}}(E/\mathbb{R},\mathbb{R}(3)) = H^1(E/\mathbb{R},\mathbb{R}(2))$ spanned by $[\omega]+[\overline{\omega}]$ (here $\mathbb{R}(\ell)=(2\pi i)^{\ell}\mathbb{R}$). Then we may write

$$\sum_j \rho(x_j,y_j) = \alpha([\omega]+[\overline{\omega}]),$$

from which we obtain

$$\int_{\partial\Gamma_0}\sum_j \rho(x_j,y_j) = \alpha 2\operatorname{Re}(\Omega).$$

On the other hand, we have

$$\int_{E(\mathbb{C})}\sum_j \rho(x_j,y_j)\wedge\omega = \alpha\int_{E(\mathbb{C})}\overline{\omega}\wedge\omega = -\alpha.$$

Combining,

$$\int_{\partial\Gamma_0}\sum_j \rho(x_j,y_j) = -2\operatorname{Re}(\Omega)\int_{E(\mathbb{C})}\sum_j \rho(x_j,y_j)\wedge\omega.$$

By Theorem 5, this equals

$$-\frac{8}{3}\operatorname{Re}(\Omega)\sum_j \sum_{\gamma_1+\gamma_2+\gamma_3=0}{}' \frac{(P_j,\gamma_1)(Q_j,\gamma_2)(R_j,\gamma_3)(\overline{\gamma_3}-\overline{\gamma_2})}{|\gamma_1|^2|\gamma_2|^2|\gamma_3|^2}$$

and the statement follows. □

The following result, initially conjectured by Deninger and proved by Goncharov, explains the connection with the L-function.

Theorem 7 ([8, Conjecture 6.5], [11, Theorem 1.1]). *Let $E/\mathbb{Q}$ be a (modular) elliptic curve. There there are functions $x_j, y_j \in \mathbb{Q}(E)^*$ such that*

$$\sum_j x_j\wedge(1-x_j)\wedge y_j = 0 \quad \textit{in } \bigwedge^3 \mathbb{Q}(E)^*,$$

$$\sum v_z(y_j)[x_j(z)] = 0 \quad \textit{in } B_2(\overline{\mathbb{Q}}) \quad \textit{for all } z\in E(\overline{\mathbb{Q}})$$

such that

$$L(E,3) \sim_{\mathbb{Q}^*} \left(\frac{2\pi A(\Lambda)}{N_E}\right)^2 \Omega_{\mathbb{R}} \sum_j \sum_{\gamma_1+\gamma_2+\gamma_3=0}{}' \frac{(P_j,\gamma_1)(Q_j,\gamma_2)(R_j,\gamma_3)(\overline{\gamma_3}-\overline{\gamma_2})}{|\gamma_1|^2|\gamma_2|^2|\gamma_3|^2},$$

where $\Omega_{\mathbb{R}} = \int_{E(\mathbb{R})}\omega$ is the real period of E and P_j, Q_j, R_j are the divisors of $y_j, x_j, 1-x_j$ respectively.

Notice that $K_4(E)$ is conjectured to have dimension 1 (the order of vanishing of $L(E,s)$ at $s = -1$), therefore, any set of functions with the condition should yield the L-function value (possibly with coefficient 0).

The formula in the previous theorem may be then written as

$$L'(E,-1) \sim_{\mathbb{Q}^*} \frac{A(\Lambda)^2 \Omega_{\mathbb{R}}}{\pi^2} \sum_j \sideset{}{'}\sum_{\gamma_1+\gamma_2+\gamma_3=0} \frac{(P_j,\gamma_1)(Q_j,\gamma_2)(R_j,\gamma_3)(\overline{\gamma_3}-\overline{\gamma_2})}{|\gamma_1|^2|\gamma_2|^2|\gamma_3|^2}.$$

We now combine these observations with the hypothesis that $\int_{E(\mathbb{C})} \omega \wedge \overline{\omega} = 1$ and Corollary 6 in order to obtain

$$\mathrm{m}(P) - \mathrm{m}(P^*) \overset{?}{\sim}_{\mathbb{Q}} \frac{\mathrm{Re}(\Omega)}{\Omega_{\mathbb{R}}} L'(E,-1),$$

where the question mark may be removed provided that $K_4(E)$ has dimension 1. The formula above should be compared to formula (2.5). As usual, it remains the difficulty of finding the rational coefficient.

This concludes the proof of Theorem 2.

Given E, x_j, and y_j as in Theorem 7, the Eisenstein–Kronecker series is determined solely by the divisors of x_j, $1-x_j$, and y_j. In order to understand this dependance, we consider the following simplification:

$$\begin{aligned}\sideset{}{'}\sum_{\gamma_1+\gamma_2+\gamma_3=0} \frac{(P,\gamma_1)(Q,\gamma_2)(R,\gamma_3)(\overline{\gamma_3}-\overline{\gamma_2})}{|\gamma_1|^2|\gamma_2|^2|\gamma_1+\gamma_2|^2} &= \sideset{}{'}\sum_{\gamma_2,\gamma_3} \frac{(P,-\gamma_2-\gamma_3)(Q,\gamma_2)(R,\gamma_3)(\overline{\gamma_3}-\overline{\gamma_2})}{|\gamma_2+\gamma_3|^2|\gamma_2|^2|\gamma_3|^2} \\ &= \sideset{}{'}\sum_{\gamma_2,\gamma_3} \frac{(Q-P,\gamma_2)(R-P,\gamma_3)(\overline{\gamma_3}-\overline{\gamma_2})}{|\gamma_2+\gamma_3|^2|\gamma_2|^2|\gamma_3|^2}.\end{aligned}$$

The above computation motivates the definition of an operation analogous to the diamond operation from Theorem 3.

Definition 8. Let $A = \sum r_j(P_j)$, $B = \sum s_k(Q_k)$, and $C = \sum t_l(R_l)$ be divisors in E. Then

$$\diamond : (\mathrm{Div}(E) \wedge \mathrm{Div}(E)) \otimes \mathrm{Div}(E) \to \mathrm{Div}(E) \wedge \mathrm{Div}(E)/\sim,$$
$$(A \wedge B) \diamond C = \sum r_j s_k t_l (P_j - R_l, Q_k - R_l),$$

where

$$(P,Q) \sim -(-P,-Q).$$

We remark that in the above formula,

$$(P,Q) = -(Q,P).$$

We will typically apply this operation when

$$A = (f), \quad B = (1-f), \quad C = (g).$$

The Eisenstein–Kronecker series in the previous theorems is then determined by

$$((f) \wedge (1-f)) \diamond (g).$$

3. Computing the Mahler measure from the numerical point of view

In this section we describe some of Boyd's ideas for numerically computing formulas such as (1.7), (1.8), and (1.9).

The starting point for computing these formulas is the following identity due to Cassaigne and Maillot [15]. Let a, b, c be nonzero complex numbers. Then

$$\pi \mathrm{m}(az + by + c) = \begin{cases} \alpha \log |a| + \beta \log |b| + \gamma \log |c| + D\left(\left|\frac{a}{b}\right| e^{i\gamma}\right), & \triangle, \\ \pi \log \max\{|a|, |b|, |c|\}, & \text{not } \triangle, \end{cases} \tag{3.1}$$

where $\triangle$ stands for the statement that $|a|$, $|b|$, and $|c|$ are the lengths of the sides of a triangle; and α, β, and γ are the angles opposite to the sides of lengths $|a|$, $|b|$ and $|c|$ respectively (Figure 1). The dilogarithm term then codifies the volume of a hyperbolic ideal tetrahedron in $\mathbb{H}^3 \cong \mathbb{C} \times \mathbb{R}_{\geq 0}$ with basis the triangle whose sides are $|a|$, $|b|$, and $|c|$ and fourth vertex infinity.

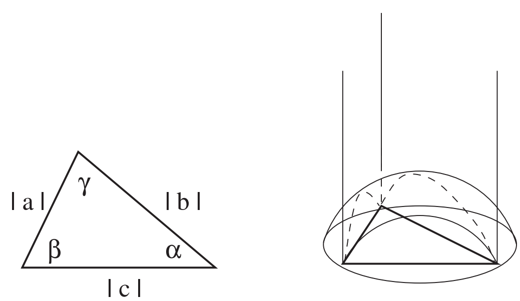

Figure 1. Relation among the parameters in Maillot's formula.

Boyd's idea consists on taking the coefficients a, b, c to be real polynomials in x. In this way, it is possible to compute

$$\mathrm{m}(a(x)z + b(x)y + c(x)) = \frac{1}{\pi} \int_0^{\pi} \mathrm{m}\Big(a(e^{i\theta})z + b(e^{i\theta})y + c(e^{i\theta})\Big) d\theta.$$

Here we have used the feature that $|a(e^{i\theta})| = |a(e^{-i\theta})|$ and likewise for b and c.

Now suppose that we take $a(x), b(x), c(x)$ to be cyclotomic polynomials. This particular class of examples is promising because of equalities such as

$$\int_a^b \log |e^{im\theta} - 1| d\theta = \frac{1}{m}\big(D(e^{imb}) - D(e^{ima})\big).$$

These terms yield algebraic numbers. In addition, $\mathrm{m}(P^*) = 0$ since P^* is certainly cyclotomic.

The main difficulty consists of evaluating the integral in the triangular case intervals. Typically, these parts of the integral have to be computed numerically. For the polynomial considered in (1.8), it is natural to split the integral in the intervals $(0, \frac{2\pi}{3})$ and $(\frac{2\pi}{3}, \pi)$. Indeed, we have the non-triangle case in the first interval and the triangle case in the second interval. Thus, it is natural to define

$$\begin{aligned} m_1 &= \frac{1}{\pi} \int_0^{\frac{2\pi}{3}} \mathrm{m}\Big(a\big(e^{i\theta}\big)z + b\big(e^{i\theta}\big)y + c\big(e^{i\theta}\big)\Big) d\theta, \\ m_2 &= \frac{1}{\pi} \int_{\frac{2\pi}{3}}^{\pi} \mathrm{m}\Big(a\big(e^{i\theta}\big)z + b\big(e^{i\theta}\big)y + c\big(e^{i\theta}\big)\Big) d\theta, \end{aligned}$$

where

$$a(x)z + b(x)y + c(x) = (x+1)z + (x^2+x+1)y + (x+1)^2.$$

This explains the meaning of the parameters m_1, m_2 in Boyd's numerical formulas (1.8)–(1.9).

4. Computing the Mahler measure from the theoretical point of view

4.1. Equation (1.7). In this subsection we consider the formula.

$$\mathrm{m}(z + (x+1)(y+1)) \stackrel{?}{=} 2L'(E_{15}, -1).$$

We will verify that this polynomial satisfies the hypothesis of Theorem 2.

We first decompose the wedge product.

$$x \wedge y \wedge z = x \wedge y \wedge (1+x)(1+y) = -x \wedge (1+x) \wedge y + y \wedge (1+y) \wedge x, \tag{4.1}$$

which will lead to integrate ρ evaluated on

$$-[-x] \otimes y + [-y] \otimes x$$

after the application of Stokes' theorem. Therefore,

$$\begin{aligned} \mathrm{m}(z + (x+1)(y+1)) &= -\frac{1}{(2\pi)^2} \int_{\Gamma_0} \eta(x, y, z) \\ &= -\frac{1}{4\pi^2} \int_{\gamma_0} -\rho(-x, y) + \rho(-y, x), \end{aligned}$$

where $\gamma_0 = \partial \Gamma_0$.

Now x and y are to be considered under the relationship resulting from eliminating z,

$$(x+1)(y+1)(x^{-1}+1)(y^{-1}+1) = 1,$$

which yields

$$(x+1)^2 y^2 + (2(x+1)^2 - x)y + (x+1)^2 = 0.$$

With the change of variables

$$x = -\frac{X}{4}, \quad y = \frac{2Y - X^2 + 6X - 16}{(X-4)^2},$$

we get a Weierstrass form

$$Y^2 = X^3 - 7X^2 + 16X. \tag{4.2}$$

The rational torsion is isomorphic to $\mathbb{Z}/4\mathbb{Z} = \langle P \rangle$, where $P = (4,4)$, $2P = (0,0)$.

We proceed to compute the diamond operation on the functions in (4.1). For that, we need to compute the divisors of all the functions involved. We first start by computing the divisors of some functions coming from the Weierstrass form

$$\begin{aligned} (X) &= 2(2P) - 2(O), \\ (X-4) &= (P) + (3P) - 2(O), \\ (2Y - X^2 + 6X - 16) &= 4(P) - 4(O), \\ (Y - X) &= 22(P) + (2P) - 3(O). \end{aligned}$$

Now we compute the divisors of the functions that appear in (4.1). Let $f_1 = -x$, $g_1 = y$, $f_2 = -y$, and $g_2 = x$. Then we have

$$\begin{aligned}
(f_1) = (g_2) = (x) &= 2(2P) - 2(O),\\
(1 - f_1) = (1 + x) &= (P) + (3P) - 2(O),\\
(g_1) = (f_2) = (y) &= 2(P) - 2(3P),\\
(1 - f_2) = (1 + y) &= \left(\frac{2(Y - X)}{(X - 4)^2}\right) = (2P) + (O) - 2(3P).
\end{aligned}$$

Combining all of the above, we obtain

$$\begin{aligned}
((f_1) \wedge (1 - f_1)) \diamond (g_1) &= 16((P, O) + (P, 2P) - (P, -P)),\\
((f_2) \wedge (1 - f_2)) \diamond (g_2) &= -16((P, O) + (P, 2P) - (P, -P)).
\end{aligned}$$

Thus, the sum coming from the wedge product (4.1) is

$$\begin{aligned}
&-((f_1) \wedge (1 - f_1)) \diamond (g_1) + ((f_2) \wedge (1 - f_2)) \diamond (g_2)\\
&\qquad = -32((P, O) + (P, 2P) - (P, -P)).
\end{aligned} \tag{4.3}$$

The above equation is one of the key results for proving Theorem 1 in the next section.

We proceed to study the integration path. We rewrite the equation

$$|(x + 1)(y + 1)| = 1$$

in polar coordinates $x = e^{i\theta}$, $y = e^{i\tau}$. Then

$$\left|4 \cos\left(\frac{\theta}{2}\right) \cos\left(\frac{\tau}{2}\right)\right| = 1.$$

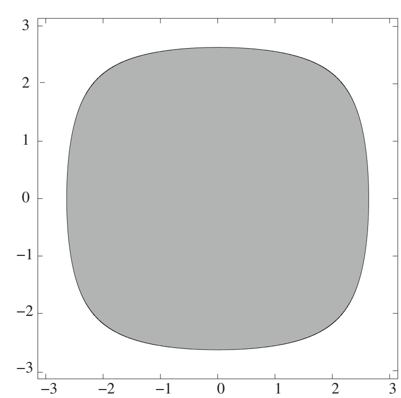

Figure 2. The shaded region indicates the integration domain.

The integration path on the unit torus is illustrated in Figure 2. The boundary γ_0 of the shaded region corresponds to a cycle in the elliptic curve. We can be more precise by observing that the holomorphic differential on the elliptic curve is given by

$$\omega = \frac{dx}{8(2(y + 1)(x + 1)^2 - x)}.$$

This formula, together with the symmetries of γ_0, implies that $\int_{\gamma_0} \omega$ is invariant by conjugation and is therefore real. Thus, $\Omega = \Omega_{\mathbb{R}}$.

In order to be able to apply Theorem 2 we need to verify that the symbols are trivial on $B_2(\overline{\mathbb{Q}})$ (this condition is analogous to the triviality of tame symbols and implies that we do get an element in $K_4(E)$, which would imply that we can expect a rational multiple of $L'(E,-1)$). It is enough to look at the valuations given by the points in the support of the divisors $(g_1),(g_2)$.

For $2P$ $(x=0,\ y=-1)$:

$$v_{2P}(g_1)[f_1(2P)]+v_{2P}(g_2)[f_2(2P)]=2[-1]=0.$$

For P $(x=-1,\ y=0)$:

$$v_P(g_1)[f_1(P)]+v_P(g_2)[f_2(P)]=2[-1]=0.$$

For $3P$ $(x=-1,\ y=\infty)$:

$$v_{3P}(g_1)[f_1(3P)]+v_{3P}(g_2)[f_2(3P)]=-2[-1]=0.$$

For O $(x=\infty,\ y=-1)$:

$$v_O(g_1)[f_1(O)]+v_O(g_2)[f_2(O)]=-2[-1]=0.$$

In sum, the conditions of Theorem 2 are verified which proves that it is correct to expect that the left hand side of (1.7) be a rational multiple of $L'(E_{15},-1)$.

4.2. Equation (1.9) and the proof of Theorem 1. We will now study the formula

$$\mathrm{m}((x+1)z+(x^2+x+1)y+(x+1)^2)\stackrel{?}{=}m_1+m_2,$$

and more specifically, relation (1.9). We will verify that this polynomial satisfies the hypothesis of Theorem 2 and we will prove Theorem 1 in the middle of this verification.

The wedge product can be decomposed in the following way.

$$\begin{aligned}(4.4)\qquad x\wedge y\wedge z&=x\wedge y\wedge\frac{(x+1)^2+(x^2+x+1)y}{x+1}\\&=x\wedge(1+x)\wedge\frac{1}{y}\left(1+\frac{(x^2+x+1)y}{(x+1)^2}\right)^2\\&\quad+\frac{(x^2+x+1)y}{(x+1)^2}\wedge\left(1+\frac{(x^2+x+1)y}{(x+1)^2}\right)\wedge x\\&\quad+x\wedge(1-x)\wedge\left(1+\frac{(x^2+x+1)y}{(x+1)^2}\right)\\&\quad-\frac{1}{3}x^3\wedge(1-x^3)\wedge\left(1+\frac{(x^2+x+1)y}{(x+1)^2}\right).\end{aligned}$$

Thus,

$$\begin{aligned}&\mathrm{m}((x+1)z+(x^2+x+1)y+(x+1)^2)\\&\quad=-\frac{1}{(2\pi)^2}\int_{\Gamma_0}\eta(x,y,z)\\&\quad=-\frac{1}{4\pi^2}\int_{\gamma_0}\rho\left(-x,\frac{1}{y}\left(1+\frac{(x^2+x+1)y}{(x+1)^2}\right)^2\right)+\rho\left(-\frac{(x^2+x+1)y}{(x+1)^2},x\right)\\&\qquad\qquad+\rho\left(x,1+\frac{(x^2+x+1)y}{(x+1)^2}\right)-\frac{1}{3}\rho\left(x^3,1+\frac{(x^2+x+1)y}{(x+1)^2}\right).\end{aligned}$$

The rational functions x and y are related by the following equation:

$$(x^2 + x + 1)((x + 1)^2 y^2 + (2(x + 1)^2 - x)y + (x + 1)^2) = 0.$$

With the change of variables

$$x = -\frac{X}{4}, \quad y = \frac{2Y - X^2 + 6X - 16}{(X - 4)^2},$$

we get the same Weierstrass form as in equation (4.2).

We proceed to compute the diamond operation in the functions in (4.4).

First we compute some divisors in the Weierstrass equation

$$\begin{aligned}
(X) &= 2(2P) - 2(O),\\
(X - 4) &= (P) + (3P) - 2(O),\\
(2Y - X^2 + 6X - 16) &= 4(P) - 4(O),\\
(Y - X) &= 2(P) + (2P) - 3(O),\\
(Y(X^2 - 4X + 16) - X(3X^2 - 20X + 48) &= 6(P) + (2P) - 7(O)
\end{aligned}$$

and

$$(X^2 - 4X + 16) = (D) + (-D) + (2P - D) + (2P + D) - 4(O).$$

In the last equation, the point D is defined by

$$D = (2(1 + \sqrt{3}i), 2(-3 + \sqrt{3}i)),$$

and it satisfies the property

$$2D = 3P.$$

Now we compute the divisors of the functions appearing in (4.4). We let

$$\begin{aligned}
&f_1 = -x, \qquad g_1 = \frac{1}{y}\left(1 + \frac{(x^2 + x + 1)y}{(x + 1)^2}\right)^2, \quad f_2 = -\frac{(x^2 + x + 1)y}{(x + 1)^2},\\
&g_2 = f_3 = x, \quad g_3 = g_4 = 1 + \frac{(x^2 + x + 1)y}{(x + 1)^2}, \quad f_4 = x^3.
\end{aligned}$$

Thus

$$\begin{aligned}
(f_1) = (f_3) = \frac{1}{3}(f_4) = (g_2) = (x) &= 2(2P) - 2(O),\\
(1 - f_1) = (1 + x) &= (P) + (3P) - 2(O),\\
(y) &= 2(P) - 2(3P),\\
(1 - f_2) = (g_3) = (g_4) &= \left(1 + \frac{(x^2 + x + 1)y}{(x + 1)^2}\right)\\
&= \left(\frac{2Y(X^2 - 4X + 16) - 2X(3X^2 - 20X + 48)}{(X - 4)^4}\right)\\
&= 2(P) + (2P) - 4(3P) + (O),\\
(g_1) &= 2(P) + 2(2P) - 6(3P) + 2(O),\\
(x^2 + x + 1) = (X^2 - 4X + 16) &= (D) + (-D) + (2P - D) + (2P + D) - 4(O),\\
(f_2) = \left(\frac{(x^2 + x + 1)y}{(x + 1)^2}\right) &= (D) + (-D) + (2P - D) + (2P + D) - 4(3P),
\end{aligned}$$

and

$$\begin{aligned}(1 - f_3) &= (1 - x) = (E) + (-E) - 2(O),\\ (1 - f_4) &= (1 - x^3) = ((X + 4)(X^2 - 4X + 16))\\ &= (E) + (-E) + (D) + (-D) + (2P - D) + (2P + D) - 6(O).\end{aligned}$$

In the last two equations the point E is given by

$$E = (-4, 4\sqrt{15}i)$$

and it verifies

$$2E = 2P.$$

Finally, the diamond operation for each term of (4.4) yields

$$\begin{aligned}((f_1) \wedge (1 - f_1)) \diamond (g_1) &= 32((P, O) + (P, 2P) - (P, -P)),\\ ((f_2) \wedge (1 - f_2)) \diamond (g_2) &= \operatorname{st} D - 16((P, O) + (P, 2P) + 2(P, -P)),\\ ((f_3) \wedge (1 - f_3)) \diamond (g_3) &= \operatorname{st} E - 24(P, -P),\\ ((f_4) \wedge (1 - f_4)) \diamond (g_4) &= 3 \operatorname{st} D + 3 \operatorname{st} E - 216(P, -P),\end{aligned}$$

where

$$\begin{aligned}\operatorname{st} D &= 24((D, -P) - (D, P) + (D + 2P, -P) - (D + 2P, P)),\\ \operatorname{st} E &= 24((P, P + E) + (P, P - E)).\end{aligned}$$

Adding all the terms from (4.4), we get

$$\begin{aligned}((f_1) \wedge (1 - f_1)) \diamond (g_1) &+ ((f_2) \wedge (1 - f_2)) \diamond (g_2)\\ &+ ((f_3) \wedge (1 - f_3)) \diamond (g_3) - \frac{1}{3}((f_4) \wedge (1 - f_4)) \diamond (g_4)\\ &= 16((P, O) + (P, 2P) - (P, -P)).\end{aligned} \tag{4.5}$$

The integration path may be written in terms of two conditions

$$x^2 + x + 1 = 0 \quad \text{or} \quad |(x + 1)(y + 1)| = 1$$

on the unit torus. In polar coordinates we get

$$\theta = \pm\frac{2\pi}{3} \quad \text{or} \quad \left|4 \cos\left(\frac{\theta}{2}\right) \cos\left(\frac{\tau}{2}\right)\right| = 1.$$

The integration set is illustrated in Figure 3. We can see two types of boundaries, a set of straight lines coming from the first condition $x^2 + x + 1 = 0$ and pieces of a cycle coming from the second condition $|(x + 1)(y + 1)| = 1$. Then m_1 represents the integral value computed in the boundary of the central region in Figure 3 and m_2 represents the integral computed on the other boundary.

Let us compute the value of $m_1 - m_2$. We obtain an integral in the cycle. By comparing equations (4.3) and (4.5), this term should lead to $-\frac{1}{2}$ the left hand side of equation (1.7). In other words, it is expected to equal $-L'(E_{15}, -1)$.

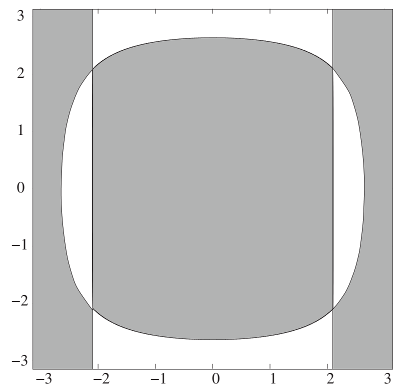

Figure 3. The shaded region indicates the integration domain.

In addition, we obtain the integral with $\theta = \frac{2\pi}{3}$ and $\pi \geq \tau \geq -\pi$ of ρ evaluated at

$$-[-\xi_3] \otimes y$$

as well as $\theta = -\frac{2\pi}{3}$ and $-\pi \leq \tau \leq \pi$ of

$$-[-\xi_3^{-1}] \otimes y,$$

where $\xi_3 = \frac{-1+\sqrt{3}i}{2}$ is the third root of unity with argument $\frac{2\pi}{3}$. This yields

$$-\frac{1}{4\pi^2}(2\pi D(-\xi_3) - 2\pi D(-\xi_3^{-1})) = -\frac{1}{\pi} D(-\xi_3) = L'(\chi_{-3}, -1).$$

Thus, we finally get

$$m_1 - m_2 = L'(\chi_{-3}, -1) - \mathrm{m}(z + (x+1)(y+1)).$$

This concludes the proof of Theorem 1.

For the Mahler measure, the computation along the straight lines yields the following:

$$-[-\xi_3] \otimes y$$

to be integrated with $-\frac{2\pi}{3} \geq \tau \geq -\pi$, $-\frac{2\pi}{3} \leq \tau \leq 0$, $\pi \geq \tau \geq 0$, and

$$-[-\xi_3^{-1}] \otimes y$$

to be integrated with $\frac{2\pi}{3} \leq \tau \leq \pi$, $\frac{2\pi}{3} \geq \tau \geq 0$, $-\pi \leq \tau \leq 0$. This yields

$$-\frac{1}{4\pi^2}\left(\frac{2\pi}{3} D(-\xi_3) - \frac{2\pi}{3} D(-\xi_3^{-1})\right) = \frac{1}{3} L'(\chi_{-3}, -1).$$

In order to completely see equation (1.8) and recover the term $\frac{13\zeta(3)}{3\pi^2}$, one should understand the integral at the level of the pieces in the cycle, but this is very difficult to do.

Finally, we may verify that we again obtain an element in $K_4(E)$ in the case of (1.9). It is enough to look at the valuations given by the points in the support of the divisors (g_i).

For P $(x = -1, y = 0)$:

$$\begin{aligned} &v_P(g_1)[f_1(P)] + v_P(g_2)[f_2(P)] + v_P(g_3)[f_3(P)] + v_P(g_4)[f_4(P)] \\ &\quad = v_P(g_1)[-1] + v_P(g_2)[0] + v_P(g_3)[-1] + v_P(g_4)[-1] = 0. \end{aligned}$$

For $2P$ $(x = 0, y = \infty)$:

$$\begin{aligned} &v_{2P}(g_1)[f_1(2P)] + v_{2P}(g_2)[f_2(2P)] + v_{2P}(g_3)[f_3(2P)] + v_{2P}(g_4)[f_4(2P)] \\ &\quad = v_P(g_1)[0] + v_P(g_2)[\infty] + v_P(g_3)[0] + v_P(g_4)[0] = 0. \end{aligned}$$

For $3P$ $(x = -1, y = \infty)$:

$$\begin{aligned} &v_{3P}(g_1)[f_1(3P)] + v_{3P}(g_2)[f_2(3P)] + v_{3P}(g_3)[f_3(3P)] + v_{3P}(g_4)[f_4(3P)] \\ &\quad = v_{3P}(g_1)[-1] + v_{3P}(g_2)[\infty] + v_{3P}(g_3)[-1] + v_{3P}(g_4)[-1] = 0. \end{aligned}$$

For O $(x = \infty, y = 0)$:

$$\begin{aligned} &v_O(g_1)[f_1(O)] + v_O(g_2)[f_2(O)] + v_O(g_3)[f_3(O)] + v_O(g_4)[f_4(O)] \\ &\quad = v_O(g_1)[\infty] + v_O(g_2)[\infty] + v_O(g_3)[\infty] + v_O(g_4)[\infty] = 0. \end{aligned}$$

This confirms that we can apply Theorem 2 and we should expect a relation with $L'(E_{15}, -1)$ on the right hand side of formula (1.9).

5. Conclusion

The calculations from the previous section prove Theorem 1, and much more. Combining the results with the discussion from Section 2, we see that equations such as (1.7) and (1.9) are to be expected, in the sense that there should be a relationship with $L'(E, -1)$ in both cases.

Two questions remind open. First, the conjecture that the dimension of $K_4(E)$ is 1, which would imply a stronger version of Theorem 2. Second, it reminds to find a method that will provide the rational coefficient of $L'(E, -1)$ in those formulas. These kinds of problems have attracted a lot of attention recently. We refer the reader to the recent work of Zudilin [22] for more information in this direction regarding formula (1.7).

Acknowledgement. The author is very grateful to David Boyd for generously sharing his numerical examples with her and for many stimulating discussions regarding this project and related ideas. The author thanks Wadim Zudilin and Mathew Rogers for several useful comments and corrections and enthusiastic discussions about this and related topics. Finally, the author is grateful to the referee for corrections and invaluable comments that have greatly improved the exposition of this paper.

References

[1] *A. A. Beĭlinson*, Higher regulators and values of L-functions of curves, Funktsional. Anal. i Prilozhen. **14** (1980), no. 2, 46–47.
[2] *M.-J. Bertin*, Mesure de Mahler d'une famille de polynômes, J. reine angew. Math. **569** (2004), 175–188.
[3] *S. J. Bloch*, Higher regulators, algebraic K-theory, and zeta functions of elliptic curves, CRM Monogr. Ser. **11**, American Mathematical Society, Providence 2000.
[4] *D. W. Boyd*, Mahler's measure and special values of L-functions, Experiment. Math. **7** (1998), 37–82.

[5] *D. W. Boyd*, Conjectural explicit formulas for the Mahler measure of some three variable polynomials, unpublished notes, 2006.
[6] *F. Brunault*, Étude de la valeur en $s = 2$ de la fonction L d'une courbe elliptique, Doctoral thesis, Université Paris 7 Denis-Diderot, Paris 2005.
[7] *F. Brunault*, Version explicite du théorème de Beilinson pour la courbe modulaire $X_1(N)$, C. R. Math. Acad. Sci. Paris **343** (2006), no. 8, 505–510.
[8] *C. Deninger*, Higher order operations in Deligne cohomology, Invent. Math. **120** (1995), no. 2, 289–315.
[9] *C. Deninger*, Deligne periods of mixed motives, K-theory and the entropy of certain $\mathbb{Z}^n$-actions, J. Amer. Math. Soc. **10** (1997), no. 2, 259–281.
[10] *A. B. Goncharov*, Geometry of configurations, polylogarithms, and motivic cohomology, Adv. Math. **114** (1995), no. 2, 197–318.
[11] *A. B. Goncharov*, Deninger's conjecture on L-functions of elliptic curves at $s = 3$, J. Math. Sci. (N. Y.) **81** (1996), no. 3, 2631–2656.
[12] *M. N. Lalín*, An algebraic integration for Mahler measure, Duke Math. J. **138** (2007), no. 3, 391–422.
[13] *M. N. Lalín*, On a conjecture by Boyd, Int. J. Number Theory **6** (2010), no. 3, 705–711.
[14] *M. N. Lalín* and *M. D. Rogers*, Functional equations for Mahler measures of genus-one curves, Algebra Number Theory **1** (2007), no. 1, 87–117.
[15] *V. Maillot*, Géométrie d'Arakelov des variétés toriques et fibrés en droites intégrables, Mém. Soc. Math. Fr. (N.S.) **80** (2000), 1–129.
[16] *V. Maillot*, Mahler measure in Arakelov geometry, workshop lecture at "The many aspects of Mahler's measure", Banff International Research Station, Banff 2003.
[17] *A. Mellit*, Elliptic dilogarithms and parallel lines, preprint 2012, `http://arxiv.org/abs/1207.4722`.
[18] *F. Rodriguez-Villegas*, Modular Mahler measures I, in: Topics in number theory (University Park 1997), Math. Appl. **467**, Kluwer Academic, Dordrecht (1999), 17–48.
[19] *F. Rodriguez-Villegas*, Identities between Mahler measures, in: Number theory for the millennium III. Proceedings of the millennial conference on number theory (Urbana-Champaign 2000), A K Peters, Natick (2002), 223–229.
[20] *M. Rogers* and *W. Zudilin*, From L-series of elliptic curves to Mahler measures, Compos. Math. **148** (2012), no. 2, 385–414.
[21] *M. Rogers* and *W. Zudilin*, On the Mahler measures of $1 + X + 1/X + Y + 1/Y$, Int. Math. Res. Not. IMRN, to appear.
[22] *W. Zudilin*, Regulator of modular units and Mahler measures, Math. Proc. Cambridge Philos. Soc., to appear.

Matilde Lalín, Département de mathématiques et de statistique, Université de Montréal,
CP 6128, succ. Centre-ville, Montreal, QC H3C 3J7, Canada
e-mail: mlalin@dms.umontreal.ca

Eingegangen 8. Mai 2013, in revidierter Fassung 24. Oktober 2013