Mahler measure and regulators

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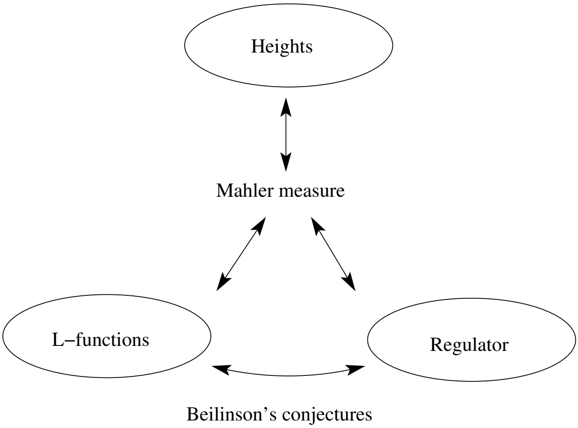
March 2nd, 2007


Heights
Mahler measure
L−functions
Regulator
Beilinson's conjectures

Mahler measure of one-variable polynomials

Pierce (1918) $P \in \mathbb{Z}[x]$ monic,

$$P(x) = \prod_i (x - \alpha_i)$$

$$\Delta_n = \prod_i (\alpha_i^n - 1)$$

$$P(x) = x - 2 \Rightarrow \Delta_n = 2^n - 1$$

Lehmer (1933)

$$\frac{\Delta_{n+1}}{\Delta_n}$$

$$\lim_{n\to\infty} \frac{|\alpha^{n+1}-1|}{|\alpha^n-1|} = \begin{cases} |\alpha| & \text{if } |\alpha|>1 \\ 1 & \text{if } |\alpha|<1 \end{cases}$$

For

$$P(x) = a\prod_i (x-\alpha_i)$$

$$M(P) = |a|\prod_i \max\{1,|\alpha_i|\}$$

$$m(P) = \log M(P) = \log|a| + \sum_i \log^+|\alpha_i|$$

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Kronecker's Lemma

$P \in \mathbb{Z}[x]$, $P \neq 0$,

$$m(P) = 0 \Leftrightarrow P(x) = x^n \prod \phi_i(x)$$

Lehmer's Question

$$m(x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1)$$

$$= 0.162357612\ldots$$

Lehmer(1933) Does there exist $C > 0$ such that $P(x) \in \mathbb{Z}[x]$

$$m(P) = 0 \quad \text{or} \quad m(P) > C??$$

$$\sqrt{\Delta_{379}} = 1,794,327,140,357$$

Mahler measure of multivariable polynomials

$P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, the (logarithmic) *Mahler measure* is :

$$\begin{aligned} m(P) &= \int_0^1 \dots \int_0^1 \log |P(e^{2\pi i\theta_1}, \dots, e^{2\pi i\theta_n})| d\theta_1 \dots d\theta_n \\ &= \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log |P(x_1, \dots, x_n)| \frac{dx_1}{x_1} \dots \frac{dx_n}{x_n} \end{aligned}$$

Jensen's formula:

$$\int_0^1 \log |e^{2\pi i\theta} - \alpha| d\theta = \log^+ |\alpha|$$

recovers one-variable case.

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Properties

- $m(P) \geq 0$ if P has integral coefficients.
- $m(P \cdot Q) = m(P) + m(Q)$
- α algebraic number, and P_α minimal polynomial over $\mathbb{Q}$,

$$m(P_\alpha) = [\mathbb{Q}(\alpha) : \mathbb{Q}]\, h(\alpha)$$

where h is the logarithmic Weil height.

Jensen's formula $\longrightarrow$ simple expression in one-variable case.

Several-variable case?

Examples in several variables

Smyth (1981)

- $$m(1+x+y) = \frac{3\sqrt{3}}{4\pi}\mathrm{L}(\chi_{-3},2) = \mathrm{L}'(\chi_{-3},-1)$$

- $$m(1+x+y+z) = \frac{7}{2\pi^2}\zeta(3)$$

$$\mathrm{L}(\chi_{-3},s) = \sum_{n=1}^{\infty}\frac{\chi_{-3}(n)}{n^s} \qquad \chi_{-3}(n) = \begin{cases} 1 & n \equiv 1 \bmod 3 \\ -1 & n \equiv -1 \bmod 3 \\ 0 & n \equiv 0 \bmod 3 \end{cases}$$

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$$

More examples in several variables

- D'Andrea & L. (2003)

$$\pi^2 m\left(\mathrm{Res}_t(x+yt+t^2, z+wt+t^2)\right)$$

$$= \pi^2 m\left(z(1-xy)^2 - (1-x)(1-y)\right) = \frac{4\sqrt{5}\zeta_{\mathbb{Q}(\sqrt{5})}(3)}{\zeta(3)}$$

- Boyd & L. (2005)

$$\pi^2 m(x^2+1+(x+1)y+(x-1)z) = \pi \mathrm{L}(\chi_{-4},2) + \frac{21}{8}\zeta(3)$$

- L. (2003)

$$\pi^3 m\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)(1+y)z\right) = 24\mathrm{L}(\chi_{-4},4)$$

- $$\pi^4 m\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)(1+y)z\right)=93\zeta(5)$$

- Known formulas for

$$\pi^{n+2} m\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_n}{1+x_n}\right)(1+y)z\right)$$

Why do we get nice numbers?

Philosophy of Beilinson's conjectures

Global information from local information through L-functions

- Arithmetic-geometric object X (for instance, $X = \mathcal{O}_F$, F a number field)
- L-function ($\mathrm{L}_F = \zeta_F$)
- Finitely-generated abelian group K ($K = \mathcal{O}_F^*$)
- Regulator map $\mathrm{reg} : K \to \mathbb{R}$ ($\mathrm{reg} = \log|\cdot|$)

$$(K \text{ rank } 1) \qquad \mathrm{L}'_X(0) \sim_{\mathbb{Q}^*} \mathrm{reg}(\xi)$$

(Dirichlet class number formula, for F real quadratic, $\zeta'_F(0) \sim_{\mathbb{Q}^*} \log|\epsilon|$, $\epsilon \in \mathcal{O}_F^*$)

An algebraic integration for Mahler measure

Deninger (1997) General framework.

Rodriguez-Villegas (1997)

$$P(x,y) = y + x - 1 \qquad X = \{P(x,y) = 0\}$$

$$m(P) = \frac{1}{(2\pi \mathrm{i})^2} \int_{\mathbb{T}^2} \log |y + x - 1| \frac{\mathrm{d}x}{x} \frac{\mathrm{d}y}{y}$$

By Jensen's equality:

$$= \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{T}^1} \log^+ |1 - x| \frac{\mathrm{d}x}{x}$$

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$$= \frac{1}{2\pi \mathrm{i}} \int_\gamma \log |y| \frac{\mathrm{d}x}{x} = -\frac{1}{2\pi \mathrm{i}} \int_\gamma \eta(x, y)$$

where

$$\gamma = X \cap \{|x| = 1, |y| \geq 1\}$$

$$\eta(x, y) = \log |x| \mathrm{d}\mathrm{i} \arg y - \log |y| \mathrm{d}\mathrm{i} \arg x$$

$$\mathrm{d} \arg x = \mathrm{Im} \left(\frac{\mathrm{d}x}{x} \right)$$

- $\eta(x,y) = -\eta(y,x)$
- $\eta(x_1x_2,y) = \eta(x_1,y) + \eta(x_2,y)$

Theorem

$$\eta(x, 1-x) = \mathrm{d}iD(x)$$

Bloch–Wigner dilogarithm:

$$D(x) := \mathrm{Im}(\mathrm{Li}_2(x)) + \arg(1-x)\log|x|$$

$$\mathrm{Li}_2(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^2} \qquad |x| < 1$$

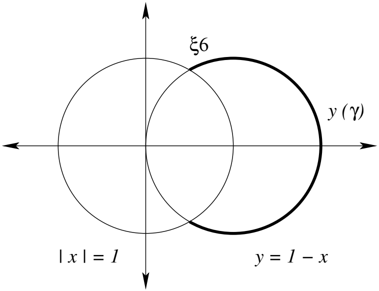
Use Stokes's Theorem:

$$m(P) = -\frac{1}{2\pi} D(\partial\gamma)$$

$x = e^{2\pi i\theta}$,

$$y(\gamma(\theta)) = 1 - e^{2\pi i\theta}, \quad \theta \in [1/6\,;5/6]$$

$$\partial\gamma = [\bar{\xi}_6] - [\xi_6]$$



$$2\pi m(x+y+1) = D(\xi_6) - D(\bar{\xi}_6)$$

$$= 2D(\xi_6) = \frac{3\sqrt{3}}{2}\mathrm{L}(\chi_{-3}, 2)$$

In general,
$P(x,y) \in \mathbb{Q}[x,y]$

$$m(P) = m(P^*) - \frac{1}{2\pi \mathrm{i}} \int_\gamma \eta(x,y)$$

$$P(x,y) = P^*(x) y^{d_y} + \dots$$

Need

$$x \wedge y = \sum_j r_j \; z_j \wedge (1 - z_j) \quad in \quad \bigwedge^2 (\mathbb{C}(X)^*) \otimes \mathbb{Q}$$

$(\{x,y\} = 0$ in $K_2(\mathbb{C}(X)) \otimes \mathbb{Q})$.

$$\int_\gamma \eta(x,y) = \sum r_j D(z_j)|_{\partial\gamma}$$

F field. Bloch group:

$$\mathcal{B}_2(F) := \mathbb{Z}[\mathbb{P}^1_F] / \langle \{0\}, \{\infty\}, R_2(x,y) \rangle$$

$$R_2(x,y) = \{x\}_2 + \{y\}_2 + \{1-xy\}_2 + \left\{ \frac{1-x}{1-xy} \right\}_2 + \left\{ \frac{1-y}{1-xy} \right\}_2$$

is the five-term relation for D.

$$\mathcal{L}_3(x) := \mathrm{Re}\left(\mathrm{Li}_3(x) - \log|x| \mathrm{Li}_2(x) + \frac{1}{3} \log^2 |x| \mathrm{Li}_1(x) \right)$$

$$\mathcal{B}_3(F) := \mathbb{Z}[\mathbb{P}^1_F] / \text{"functional equations of } \mathcal{L}_3(x)\text{"}$$

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The three-variable case

Theorem

L. (2005)
$P(x,y,z) \in \mathbb{Q}[x,y,z]$ *irreducible, nonreciprocal,*

$$X = \{P(x,y,z) = 0\}, \qquad C = \{\mathrm{Res}_z(P(x,y,z), P(x^{-1},y^{-1},z^{-1})) = 0\}$$

$$x \wedge y \wedge z = \sum_i r_i x_i \wedge (1-x_i) \wedge y_i \qquad in \qquad \bigwedge^3 (\mathbb{C}(X)^*) \otimes \mathbb{Q},$$

$$\{x_i\}_2 \otimes y_i = \sum_j r_{i,j} \{x_{i,j}\}_2 \otimes x_{i,j} \qquad in \qquad (\mathcal{B}_2(\mathbb{C}(C)) \otimes \mathbb{C}(C)^*)_{\mathbb{Q}}$$

Then

$$4\pi^2(m(P^*) - m(P)) = \mathcal{L}_3(\xi) \qquad \xi \in \mathcal{B}_3(\bar{\mathbb{Q}})_{\mathbb{Q}}$$

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$$\{x,y,z\} = 0 \quad \textit{in} \quad K_3^M(\mathbb{C}(X)) \otimes \mathbb{Q}$$

$$\{x_i\}_2 \otimes y_i \quad \textit{trivial in} \quad gr_3^\gamma K_4(\mathbb{C}(C)) \otimes \mathbb{Q}\,(?)$$

Then

$$4\pi^2(m(P^*) - m(P)) = \mathcal{L}_3(\xi) \qquad \xi \in \mathcal{B}_3(\bar{\mathbb{Q}})_{\mathbb{Q}}$$

- Explains all the known cases involving $\zeta(3)$ (by Borel's Theorem).
- It is constructive (no need of "happy idea" integrals).
- Integration sets hard to describe.
- Conjecture for n-variables using Goncharov's regulator currents. Provides motivation for Goncharov's construction.

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The measures of a family of genus-one curves

$$m(k) := m\left(x + \frac{1}{x} + y + \frac{1}{y} + k\right)$$

Boyd (1998)

$$m(k) \stackrel{?}{=} \frac{\mathrm{L}'(E_k, 0)}{s_k} \quad k \in \mathbb{N} \neq 0, 4$$

E_k determined by $x + \frac{1}{x} + y + \frac{1}{y} + k = 0$.

Rogers & L (2006)
For $|h| < 1$, $h \neq 0$,

$$m\left(2\left(h+\frac{1}{h}\right)\right) + m\left(2\left(\mathrm{i}h+\frac{1}{\mathrm{i}h}\right)\right) = m\left(\frac{4}{h^2}\right).$$

Kurokawa & Ochiai (2005)
For $h \in \mathbb{R}^*$,

$$m(4h^2) + m\left(\frac{4}{h^2}\right) = 2m\left(2\left(h+\frac{1}{h}\right)\right).$$

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$h = \frac{1}{\sqrt{2}}$ in both equations, and some K-theory,

Corollary

$$m(8) = 4m(2) = \frac{8}{5} m\left(3\sqrt{2}\right)$$

Rodriguez-Villegas (1997)

$k = 3\sqrt{2}$ (modular curve $X_0(24)$)

$$m\left(3\sqrt{2}\right) = m\left(x + \frac{1}{x} + y + \frac{1}{y} + 3\sqrt{2}\right) = q\mathrm{L}'(E_{3\sqrt{2}}, 0)$$

$$q \in \mathbb{Q}^*, \quad q \stackrel{?}{=} \frac{5}{2}$$

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$$q \in \mathbb{Q}^*, \quad q \stackrel{?}{=} \frac{5}{2}$$

For $|k| > 4$, $x + \frac{1}{x} + y + \frac{1}{y} + k$ does not intersect $\mathbb{T}^2$.

$$m(k) = -\frac{1}{2\pi \mathrm{i}} \int_{\gamma} \eta(x, y)$$

where

$$\gamma = X \cap \{|x| = 1\}$$

$$\eta(x, y) = \log |x| \mathrm{d} \mathrm{i} \arg y - \log |y| \mathrm{d} \mathrm{i} \arg x$$

We are evaluating the regulator in $\{x, y\} \in K_2(E)_{\mathbb{Q}}$.

Computing the regulator

$$E(\mathbb{C}) \cong \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z} \cong \mathbb{C}^*/q^{\mathbb{Z}}$$

$z \bmod \Lambda = \mathbb{Z} + \tau\mathbb{Z}$ is identified with $\mathrm{e}^{2\mathrm{i}\pi z}$.
Bloch regulator function

$$R_\tau\left(\mathrm{e}^{2\pi\mathrm{i}(a+b\tau)}\right) = \frac{y_\tau^2}{\pi} \sideset{}{'}\sum_{m,n\in\mathbb{Z}} \frac{\mathrm{e}^{2\pi\mathrm{i}(bn-am)}}{(m\tau+n)^2(m\bar{\tau}+n)}$$

y_τ is the imaginary part of τ.

Theorem

L. & Rogers (2006), after results of Beilinson, Bloch, idea of Deninger

$E/\mathbb{R}$ elliptic curve, x, y are non-constant functions in $\mathbb{C}(E)$ with trivial tame symbols, $\omega \in \Omega^1$

$$-\int_\gamma \eta(x,y) = \mathrm{Im}\left(\frac{\Omega}{y_\tau \Omega_0} R_\tau\left((x) \diamond (y)\right)\right)$$

where Ω_0 is the real period and $\Omega = \int_\gamma \omega$.

In our case,

$$\mathbb{Z}[E(\mathbb{C})]^- \ni (x) \diamond (y) = 8(P), \qquad P \quad \text{4-torsion.}$$

Isogenies $\rightsquigarrow$ Functional eq for the regulator.

Functional eq for the regulator $\rightsquigarrow$ Functional eq for the Mahler measure

Big picture

$$\cdots \to (K_3(\bar{\mathbb{Q}}) \supset) K_3(\partial\gamma) \to K_2(X, \partial\gamma) \to K_2(X) \to \dots$$

$$\partial\gamma = X \cap \mathbb{T}^2$$

- $\eta(x, y)$ is exact, then $\{x, y\} \in K_3(\partial\gamma)$. We have $\partial\gamma \neq \emptyset$ and we use Stokes's Theorem.
  $\rightsquigarrow D$, $1 + x + y$

- $\partial\gamma = \emptyset$, then $\{x, y\} \in K_2(C)$. We have $\eta(x, y)$ is not exact.
  $\rightsquigarrow L$-function, $1 + x + \frac{1}{x} + y + \frac{1}{y}$

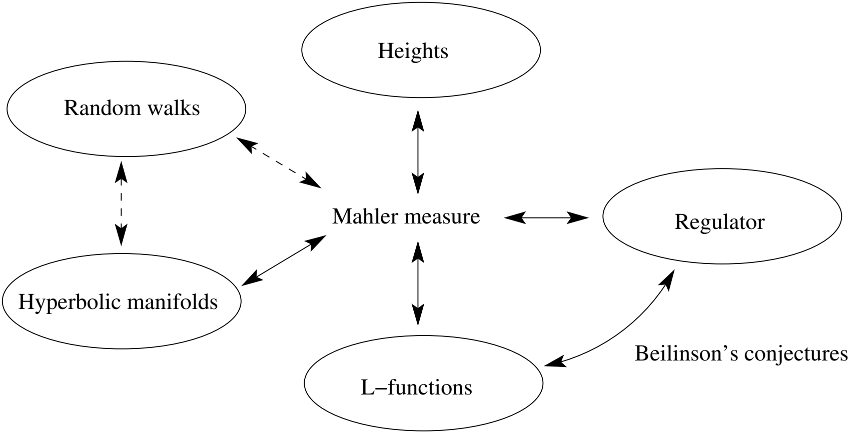
Big picture in three variables

$$\cdots \to K_4(\partial\Gamma) \to K_3(X, \partial\Gamma) \to K_3(X) \to \dots$$

$$\partial\Gamma = X \cap \mathbb{T}^3$$

$$\cdots \to (K_5(\bar{\mathbb{Q}}) \supset) K_5(\partial\gamma) \to K_4(C, \partial\gamma) \to K_4(C) \to \dots$$

$$\partial\gamma = C \cap \mathbb{T}^2$$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