# Mahler measure under variations of the base group

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# Mahler measure of several variable polynomials

$P \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, the (logarithmic) *Mahler measure* is :

$$\begin{aligned} m(P) &= \int_0^1 \ldots \int_0^1 \log |P(\mathrm{e}^{2\pi \mathrm{i}\theta_1}, \ldots, \mathrm{e}^{2\pi \mathrm{i}\theta_n})| \mathrm{d}\theta_1 \ldots \mathrm{d}\theta_n \\ &= \frac{1}{(2\pi \mathrm{i})^n} \int_{\mathbb{T}^n} \log |P(x_1, \ldots, x_n)| \frac{\mathrm{d}x_1}{x_1} \ldots \frac{\mathrm{d}x_n}{x_n} \end{aligned}$$

By Jensen's formula,

$$m\left(a \prod (x - \alpha_i)\right) = \log |a| + \sum \log \max\{1, |\alpha_i|\}.$$

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# Examples in several variables

- Smyth (1981)

$$m(1+x+y) = \frac{3\sqrt{3}}{4\pi}L(\chi_{-3},2) = \frac{\mathrm{Vol(Fig8)}}{2\pi}$$

- Boyd, Deninger, Rodriguez-Villegas (1997)

$$m\left(x+\frac{1}{x}+y+\frac{1}{y}-1\right) \stackrel{?}{=} L'(E_1,0)$$

$E_1$ elliptic curve, projective closure of $x+\frac{1}{x}+y+\frac{1}{y}-1=0$.
(50 decimal places)

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# The general technique

Rodriguez-Villegas (1997)

$$P_\lambda(x,y) = 1 - \lambda P(x,y) \qquad P(x,y) = x + \frac{1}{x} + y + \frac{1}{y}$$

Reciprocal

$$m(P,\lambda) := m(P_\lambda)$$

$$m(P,\lambda) = \frac{1}{(2\pi \mathrm{i})^2} \int_{\mathbb{T}^2} \log|1 - \lambda P(x,y)| \frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y}.$$

Note

$$|\lambda P(x,y)| < 1, \qquad \lambda \quad \text{small}, \quad x,y \in \mathbb{T}^2$$

$$\tilde{m}(P,\lambda) = \frac{1}{(2\pi \mathrm{i})^2}\int_{\mathbb{T}^2} \log(1-\lambda P(x,y))\frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y}$$

$$= -\sum_{n=1}^{\infty}\frac{\lambda^n}{n}\frac{1}{(2\pi \mathrm{i})^2}\int_{\mathbb{T}^2} P(x,y)^n \frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y} = -\sum_{n=1}^{\infty}\frac{a_n\lambda^n}{n}$$

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Let

$$u(P,\lambda) = \frac{1}{(2\pi \mathrm{i})^2}\int_{\mathbb{T}^2} \frac{1}{1-\lambda P(x,y)}\frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y} = \sum_{n=0}^{\infty} a_n \lambda^n$$

$$\frac{\mathrm{d}\tilde{m}(P,\lambda)}{\mathrm{d}\lambda} = -\frac{1}{(2\pi \mathrm{i})^2}\int_{\mathbb{T}^2} \frac{P(x,y)}{1-\lambda P(x,y)}\frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y}$$

In the case $P = x + \frac{1}{x} + y + \frac{1}{y}$,

$$a_n = \begin{cases} \binom{2m}{m}^2 & n = 2m \\ \\ 0 & \textit{otherwise} \end{cases}$$

## Definition

$\Gamma$ *finitely generated group with generators* $x_1, \dots, x_l$

$$Q = Q(x_1, \dots, x_l) = \sum_{g \in \Gamma} c_g g \in \mathbb{C}\Gamma,$$

$$Q^* = \sum_{g \in \Gamma} \overline{c_g} g^{-1} \in \mathbb{C}\Gamma \text{ reciprocal.}$$

$P = P(x_1, \dots, x_l) \in \mathbb{C}\Gamma$ , $P = P^*$, $|\lambda|^{-1} >$ *length of* $P$,

$$m_\Gamma(P, \lambda) = -\sum_{n=1}^{\infty} \frac{a_n \lambda^n}{n},$$

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We also write

$$u_\Gamma(P,\lambda)=\sum_{n=0}^{\infty} a_n\lambda^n$$

for the generating function of the $a_n$.

$Q(x_1,\ldots,x_l)\in\mathbb{C}\Gamma$

$$QQ^*=\frac{1}{\lambda}\left(1-(1-\lambda QQ^*)\right)$$

for $\lambda$ real and positive and $1/\lambda$ larger than the length of $QQ^*$.

$$m_\Gamma(Q)=-\frac{\log\lambda}{2}-\sum_{n=1}^{\infty}\frac{b_n}{2n},\qquad b_n=\left[(1-\lambda QQ^*)^n\right]_0.$$

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# Lück's combinatorial $L^2$-torsion.

$K$ knot

$$\Gamma = \pi_1(S^3 \setminus K) = \langle x_1, \dots, x_g \mid r_1, \dots, r_{g-1} \rangle$$

Let

$$F = \begin{pmatrix} \frac{\partial r_1}{\partial x_1} & \cdots & \frac{\partial r_1}{\partial x_g} \\ \vdots & \ddots & \vdots \\ \frac{\partial r_{g-1}}{\partial x_1} & \cdots & \frac{\partial r_{g-1}}{\partial x_g} \end{pmatrix} \in M^{(g-1)\times g}(\mathbb{C}\Gamma)$$

Fox matrix.

$$D(u+v) = D(u) + D(v)$$

$$D(u \cdot v) = D(u)\epsilon(v) + uD(v)$$

Delete a column $F \rightsquigarrow A \in M^{(g-1)\times(g-1)}(\mathbb{C}\Gamma)$.

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**Theorem (Lück, 2002)**

*Suppose $K$ is a hyperbolic knot. Then, for $\lambda$ sufficiently large*

$$\frac{1}{3\pi}\mathrm{Vol}(K) = -(g-1)\ln\lambda - \sum_{n=1}^{\infty}\frac{1}{n}\mathrm{tr}_{\mathbb{C}\Gamma}\left((1-\lambda AA^*)^n\right).$$

# Cayley Graphs

$\Gamma$ of order $m$

$$\alpha : \Gamma \to \mathbb{C} \qquad \alpha(g) = \overline{\alpha(g^{-1})} \quad \forall g \in \Gamma$$

Weighted Cayley graph:

- Vertices $g_1, \ldots, g_m$.
- (directed) Edge between $g_i$ and $g_j$ has weight $\alpha(g_i^{-1} g_j)$.

Weighted adjacency matrix

$$A(\Gamma, \alpha) = \{\alpha(g_i^{-1} g_j)\}_{i,j}$$

# The Mahler measure over finite groups

$\Gamma$

$$P \in \mathbb{C}\Gamma$$

reciprocal

Assume monomials generate $\Gamma$.

**Theorem**

*For $\Gamma$ finite*

$$m_\Gamma(P,\lambda) = \frac{1}{|\Gamma|}\log\det(I - \lambda A),$$

*$A$ is the adjacency matrix of the Cayley graph (with weights) and $\frac{1}{\lambda} > \rho(A)$.*

Analytic continuation for $m_\Gamma(P,\lambda)$ to $\mathbb{C} \setminus \mathrm{Spec}(A)$.

# Finite Abelian Groups

$$\Gamma = \mathbb{Z}/m_1\mathbb{Z} \times \cdots \times \mathbb{Z}/m_l\mathbb{Z}$$

Corollary

$$m_\Gamma(P,\lambda) = \frac{1}{|\Gamma|} \log \left( \prod_{j_1,\dots,j_l} \left(1 - \lambda P(\xi_{m_1}^{j_1}, \dots, \xi_{m_l}^{j_l})\right) \right)$$

*where $\xi_k$ is a primitive root of unity.*

Babai (1979): Spectra of Cayley graph is related to irreducible characters of $\Gamma$.

# Approximations

## Proposition

*For small $\lambda$,*

- $$\lim_{m_1,\dots,m_l\to\infty} m_{\mathbb{Z}/m_1\mathbb{Z}\times\cdots\times\mathbb{Z}/m_l\mathbb{Z}}(P,\lambda) = m_{\mathbb{Z}^l}(P,\lambda).$$

  *Where the limit is with $m_1,\dots,m_l$ going to infinity independently.*
- *For $\Gamma = D_\infty$, $\Gamma_m = D_m$,*

  $$\lim_{m\to\infty} m_{D_m}(P,\lambda) = m_{D_\infty}(P,\lambda).$$
- *For $\Gamma = PSL_2(\mathbb{Z}) = \langle x,y \,|\, x^2, y^3\rangle$, $\Gamma_m = \langle x,y \,|\, x^2, y^3, (xy)^m\rangle$,*

  $$\lim_{m\to\infty} m_{\Gamma_m}(P,\lambda) = m_{PSL_2(\mathbb{Z})}(P,\lambda).$$

# $x + x^{-1} + y + y^{-1}$ revisited

Now $P = x + x^{-1} + y + y^{-1}$.

$$u_{\mathbb{Z}\times\mathbb{Z}}(P,\lambda) = \sum_{n=0}^{\infty} \binom{2n}{n}^2 \lambda^{2n} = {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;16\lambda^2\right)$$

$$u_{\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}}(P,\lambda) = \sum_{n=0}^{\infty} \binom{4n}{2n} \lambda^{2n}$$

$$u_{\mathbb{Z}*\mathbb{Z}}(P,\lambda) = \frac{3}{1+2\sqrt{1-12\lambda^2}}$$

Generating function of the circuits of a $d$-regular tree (Bartholdi, 1999).

$$g_d(\lambda) = \frac{2(d-1)}{d-2+d\sqrt{1-4(d-1)\lambda^2}}.$$

$$x_1 + x_1^{-1} + \cdots + x_l + x_l^{-1}$$

$$(1 + x_1 + \cdots + x_{l-1})(1 + x_1^{-1} + \cdots + x_{l-1}^{-1})$$

# Recurrence relations $x + x^{-1} + y + y^{-1}$

Coefficients satisfy recurrence relations

$$\mathbb{Z} \times \mathbb{Z}: \quad n^2 a_{2n} - 4(2n-1)^2 a_{2n-2} = 0$$

$$\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}: \quad n(2n-1) a_{2n} - 2(4n-1)(4n-3) a_{2n-2} = 0$$

$$\mathbb{Z} * \mathbb{Z}: \quad n a_{2n} - 2(14n-9) a_{2n-2} + 96(2n-3) a_{2n-4} = 0$$

- $\mathbb{Z}^l$

  Rodriguez - Villegas: $u(\lambda)$ periods of a holomorphic differential in the curve defined by $1 = \lambda P(x, y)$.By Griffiths (1969)

$$A(\lambda)u'' + B(\lambda)u' + C(\lambda)u = 0,$$

  Picard-Fuchs differential equation ($A$, $B$, $C$ polynomials).
  $\Rightarrow$ Recurrence of the coefficients.
- This recurrence result extends to the case of $\Gamma$ finitely generated abelian group.
- $\mathbb{F}_l$

  By Haiman (1993): $u(\lambda)$ is algebraic.
  Algebraic functions in non-commuting variables.

$P = x + x^{-1} + y + y^{-1}$

$$\Gamma = \left\langle x, y \,|\, x^2 y = y x^2, y^2 x = x y^2 \right\rangle$$

Domb (1960)

$$a_{2n} = \sum_{k=0}^{n} \binom{n}{k} \binom{2k}{k} \binom{2n-2k}{n-k}$$

Same as ordinary Mahler measure for

$$1 - \lambda \left(x + x^{-1} + z \left(y + y^{-1}\right)\right) \left(x + x^{-1} + z^{-1} \left(y + y^{-1}\right)\right)$$

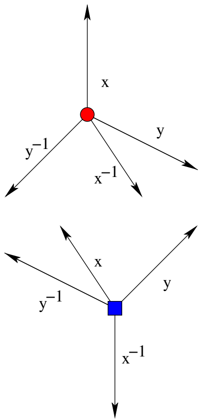
$$n^3 a_{2n} - 2(2n-1)(5n^2-5n+2)a_{2n-2} + 6(n-1)^3 a_{2n-4} = 0$$

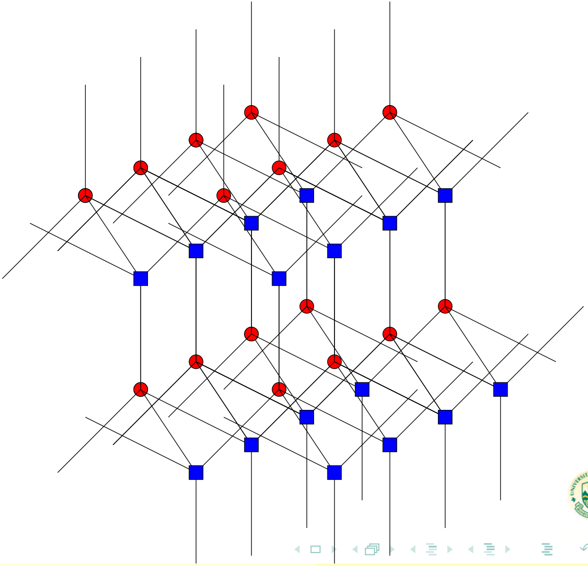
Rogers (2007)

$$1-\lambda\left(4+\left(x+x^{-1}\right)\left(y+y^{-1}\right)+\left(y+y^{-1}\right)\left(z+z^{-1}\right)+\left(z+z^{-1}\right)\left(x+x^{-1}\right)\right)$$

$${}_3F_2\left(\frac{1}{3},\frac{1}{2},\frac{2}{3};1,1;-\frac{108\lambda}{(1-16\lambda)^3}\right) = (1-16\lambda)\sum_{n=0}^{\infty} a_{2n}\lambda^n$$

# The diamond lattice



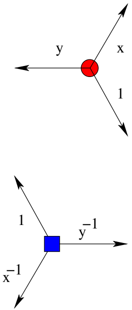



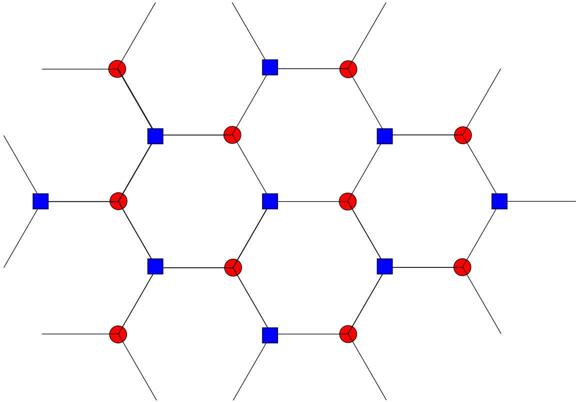
$$Q = (1 + x + y)\left(1 + x^{-1} + y^{-1}\right)$$

$$[Q^n]_0 = a_n$$

$$n^2 a_n - (10n^2 - 10n + 3)a_{n-1} + 9(n-1)^2 a_{n-2} = 0,$$

# Honeycomb lattice $(1 + x + y)\left(1 + x^{-1} + y^{-1}\right)$



$$P = x + x^{-1} + y + y^{-1} + xy^{-1} + x^{-1}y$$

$$[P^n]_0 = b_n$$

$$n^2 b_n - n(n-1)b_{n-1} - 24(n-1)^2 b_{n-2} - 36(n-2)(n-1)b_{n-3} = 0.$$

$$Q = 3 + P$$

$$b_n = \sum_{j=0}^{n} \binom{n}{j} (-3)^{n-j} a_j$$

## Triangular lattice $x + x^{-1} + y + y^{-1} + xy^{-1} + x^{-1}y$