Hyperbolic volumes and zeta values
An introduction

Matilde N. Lalín

University of Alberta
mlalin@math.ulberta.ca
http://www.math.ualberta.ca/~mlalin

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The hyperbolic space

Hyperbolic Geometry: Lobachevsky, Bolyai, Gauss ($\sim$ 1830)

Beltrami's Half-space model (1868)

$$\mathbb{H}^n = \{(x_1, \dots, x_{n-1}, x_n) \,|\, x_i \in \mathbb{R}, x_n > 0\},$$

$$\mathrm{d}s^2 = \frac{\mathrm{d}x_1^2 + \cdots + \mathrm{d}x_n^2}{x_n^2},$$

$$\mathrm{d}V = \frac{\mathrm{d}x_1 \dots \mathrm{d}x_n}{x_n^n},$$

$$\partial\mathbb{H}^n = \{(x_1, \dots, x_{n-1}, 0)\} \cup \infty.$$

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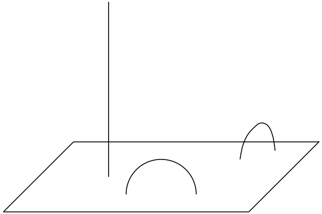
Geodesics are given by vertical lines and semicircles whose endpoints lie in $\{x_n = 0\}$ and intersect it orthogonally.



Poincaré (1882):
Orientation preserving isometries of $\mathbb{H}^2$

$$PSL(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(2,\mathbb{R}) \,\middle|\, ad - bc = 1 \right\} / \pm I.$$

$$z = x_1 + x_2 i \to \frac{az+b}{cz+d}.$$

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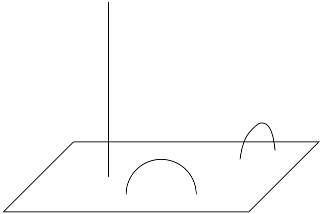

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Orientation preserving isometries of $\mathbb{H}^3$ is $PSL(2,\mathbb{C})$.

$$\mathbb{H}^3 = \{z = x_1 + x_2 i + x_3 j \,|\, x_3 > 0\},$$

subspace of quaternions ($i^2 = j^2 = k^2 = -1$, $ij = -ji = k$).

$$z \to (az+b)(cz+d)^{-1} = (az+b)(\bar{z}\bar{c}+\bar{d})|cz+d|^{-2}.$$

Poincaré: study of discrete groups of hyperbolic isometries.

Picard (1884): fundamental domain for $PSL(2,\mathbb{Z}[i])$ in $\mathbb{H}^3$ has a finite volume.
Humbert (1919) extended this result.

Volumes in $\mathbb{H}^3$

Lobachevsky function:

$$л(\theta) = -\int_0^\theta \log|2\sin t|\mathrm{d}t.$$

$$л(\theta) = \frac{1}{2}\mathrm{Im}\left(\mathrm{Li}_2\left(e^{2i\theta}\right)\right),$$

where

$$\mathrm{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}, \qquad |z| \leq 1.$$

$$\mathrm{Li}_2(z) = -\int_0^z \log(1-x)\frac{\mathrm{d}x}{x}.$$

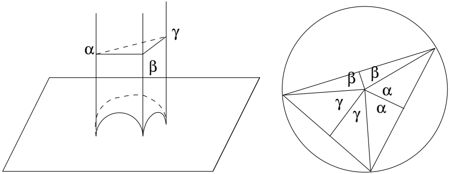
(multivalued) analytic continuation to $\mathbb{C} \setminus [1, \infty)$

Let Δ be an ideal tetrahedron (vertices in $\partial\mathbb{H}^3$).

Theorem

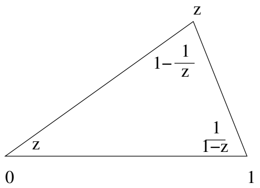
(Milnor, after Lobachevsky)
The volume of an ideal tetrahedron with dihedral angles α, β, and γ is given by

$$\mathrm{Vol}(\Delta) = л(\alpha) + л(\beta) + л(\gamma).$$



Move a vertex to ∞ and use baricentric subdivision to get six simplices with three right dihedral angles.

Triangle with angles α, β, γ, defined up to similarity.
Let $\Delta(z)$ be the tetrahedron determined up to transformations by any of $z, 1-\frac{1}{z}, \frac{1}{1-z}$.



If ideal vertices are z_1, z_2, z_3, z_4,

$$z = [z_1 : z_2 : z_3 : z_4] = \frac{(z_3 - z_2)(z_4 - z_1)}{(z_3 - z_1)(z_4 - z_2)}.$$

Bloch-Wigner dilogarithm

$$D(z) = \mathrm{Im}(\mathrm{Li}_2(z) + \log|z|\log(1-z)).$$

Continuous in $\mathbb{P}^1(\mathbb{C})$, real-analytic in $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$.

$$D(z) = -D(1-z) = -D\left(\frac{1}{z}\right) = -D(\bar{z}).$$

$$\mathrm{Vol}(\Delta(z)) = D(z).$$

Five points in $\partial\mathbb{H}^3 \cong \mathbb{P}^1(\mathbb{C})$, then the sum of the signed volumes of the five possible tetrahedra must be zero:

$$\sum_{i=0}^{5}(-1)^i \mathrm{Vol}([z_1 : \cdots : \hat{z}_i : \cdots : z_5]) = 0.$$

Five-term relation

$$D(x) + D(1-xy) + D(y) + D\left(\frac{1-y}{1-xy}\right) + D\left(\frac{1-x}{1-xy}\right) = 0.$$

Dedekind ζ-function

F number field, $[F:\mathbb{Q}] = n = r_1 + 2r_2$

$\tau_1, \ldots, \tau_{r_1}$ real embeddings

$\sigma_1, \ldots, \sigma_{r_2}$ a set of complex embeddings (one for each pair of conjugate embeddings).

$$\zeta_F(s) = \sum_{\mathfrak{A}\,\mathrm{ideal}\,\neq 0} \frac{1}{N(\mathfrak{A})^s}, \qquad \mathrm{Re}\, s > 1,$$

$N(\mathfrak{A}) = |\mathcal{O}_F/\mathfrak{A}|$ norm.

Euler product

$$\prod_{\mathfrak{P}\,\mathrm{prime}} \frac{1}{1 - N(\mathfrak{P})^{-s}}.$$

Theorem

(Dirichlet's class number formula) $\zeta_F(s)$ extends meromorphically to $\mathbb{C}$ with only one simple pole at $s = 1$ with

$$\lim_{s\to 1}(s-1)\zeta_F(s) = \frac{2^{r_1}(2\pi)^{r_2} h_F \operatorname{reg}_F}{\omega_F\sqrt{|D_F|}},$$

where

- h_F *is the class number.*
- ω_F *is the number of roots of unity in* F.
- reg_F *is the regulator.*

$$\lim_{s\to 0} s^{1-r_1-r_2}\zeta_F(s) = -\frac{h_F \operatorname{reg}_F}{\omega_F}.$$

Regulator

$\{u_1, \ldots, u_{r_1+r_2-1}\}$ basis for $\mathcal{O}_F^*$ modulo torsion

$$L(u_i) := (\log|\tau_1 u_i|, \ldots, \log|\tau_{r_1} u_i|, 2\log|\sigma_1 u_i|, \ldots, 2\log|\sigma_{r_2-1} u_i|)$$

reg_F is the determinant of the matrix.

$=$ (up to a sign) the volume of fundamental domain for $L(\mathcal{O}_F^*)$.

Euler:

$$\zeta(2m) = \frac{(-1)^{m-1}(2\pi)^{2m}B_m}{2(2m)!}$$

Klingen , Siegel:
F is totally real ($r_2 = 0$),

$$\zeta_F(2m) = r(m)\sqrt{|D_F|}\pi^{2mn}, \qquad m > 0$$

where $r(m) \in \mathbb{Q}$.

Building manifolds

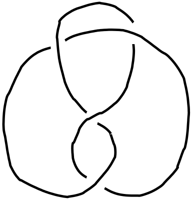
Bianchi:

- $F = \mathbb{Q}\left(\sqrt{-d}\right)$ $d \geq 1$ square-free
- Γ a torsion-free subgroup of $PSL(2, \mathcal{O}_d)$,
- $[PSL(2, \mathcal{O}_d) : \Gamma] < \infty$.

Then $\mathbb{H}^3/\Gamma$ is an oriented hyperbolic three-manifold.

Example:

$$d = 3, \quad \mathcal{O}_3 = \mathbb{Z}[\omega], \quad \omega = \frac{-1 + \sqrt{-3}}{2}$$



Riley:

$$[PSL(2, \mathcal{O}_3) : \Gamma] = 12$$

$\mathbb{H}^3/\Gamma$ diffeomorphic to $S^3 \setminus \mathrm{Fig} - 8$.

Theorem

(Essentially Humbert)

$$Vol\left(\mathbb{H}^3/PSL(2,\mathcal{O}_d)\right) = \frac{D_d\sqrt{D_d}}{4\pi^2}\zeta_{\mathbb{Q}(\sqrt{-d})}(2).$$

$$D_d = \begin{cases} d & d \equiv 3 \bmod 4, \\ 4d & \text{otherwise.} \end{cases}$$

M hyperbolic 3-manifold

$$\mathrm{Vol}(M) = \sum_{j=1}^{J} D(z_j).$$

$$\zeta_{\mathbb{Q}(\sqrt{-d})}(2) = \frac{D_d\sqrt{D_d}}{2\pi^2}\sum_{j=1}^{J} D(z_j).$$

Example:

$$\mathrm{Vol}(S^3 \setminus \mathrm{Fig}-8) = 12\frac{3\sqrt{3}}{4\pi^2}\zeta_{\mathbb{Q}(\sqrt{-3})}(2)$$

$$= 3D\left(\mathrm{e}^{\frac{2i\pi}{3}}\right) = 2D\left(\mathrm{e}^{\frac{i\pi}{3}}\right).$$

Zagier (1986):

- $[F:\mathbb{Q}] = r_1 + 2$
  Γ torsion free subgroup of finite index of the group of units of an order in a quaternion algebra B over F that is ramified at all real places.

$$\mathrm{Vol}(\mathbb{H}^3/\Gamma) \sim_{\mathbb{Q}^*} \frac{\sqrt{|D_F|}}{\pi^{2(n-1)}}\zeta_F(2).$$

- $[F:\mathbb{Q}] = r_1 + 2r_2, \qquad r_2 > 1$
  Γ discrete subgroup of $PSL(2,\mathbb{C})^{r_2}$ such that

$$\mathrm{Vol}\left(\left(\mathbb{H}^3\right)^{r_2}/\Gamma\right) \sim_{\mathbb{Q}^*} \frac{\sqrt{|D_F|}}{\pi^{2(r_1+r_2)}}\zeta_F(2).$$

$$\left(\mathbb{H}^3\right)^{r_2}/\Gamma = \bigcup \Delta(z_1) \times \cdots \times \Delta(z_{r_2})$$

The Bloch group

$$\mathrm{Vol}(M) = \sum_{j=1}^{J} D(z_j),$$

then

$$\sum_{j=1}^{J} z_j \wedge (1 - z_j) = 0 \in \bigwedge^2 \mathbb{C}^*.$$

$$\bigwedge^2 \mathbb{C}^* = \{x \wedge y \mid x \wedge x = 0,\ x_1 x_2 \wedge y = x_1 \wedge y + x_2 \wedge y\}$$

$\mathrm{Vol}(M) = D(\xi_M)$, where $\xi_M \in \mathcal{A}(\bar{\mathbb{Q}})$, and

$$\mathcal{A}(F) = \left\{ \sum n_i [z_i] \in \mathbb{Z}[F] \,\middle|\, \sum n_i z_i \wedge (1 - z_i) = 0 \right\}.$$

Let

$$\mathcal{C}(F) = \left\{ [x] + [1-xy] + [y] + \left[\frac{1-y}{1-xy}\right] + \left[\frac{1-x}{1-xy}\right] \middle| \right.$$

$$\left. x, y \in F, xy \neq 1 \right\},$$

Bloch group is

$$\mathcal{B}(F) = \mathcal{A}(F)/\mathcal{C}(F).$$

$D : \mathcal{B}(\mathbb{C}) \to \mathbb{R}$ well-defined function,
$\mathrm{Vol}(M) = D(\xi_M)$ for some $\xi_M \in \mathcal{B}(\bar{\mathbb{Q}})$, independently of the triangulation.
Then
$\zeta_F(2) = \sqrt{|D_F|}\pi^{2(n-1)} D(\xi_M)$ for $r_2 = 1$.

Theorem

(Zagier, Bloch, Suslin) For a number field $[F:\mathbb{Q}] = r_1 + 2r_2$,

- $\mathcal{B}(F)$ *is finitely generated of rank* r_2.
- $\xi_1, \ldots \xi_{r_2}$ $\mathbb{Q}$*-basis of* $\mathcal{B}(F) \otimes \mathbb{Q}$. *Then*

$$\zeta_F(2) \sim_{\mathbb{Q}^*} \sqrt{|D_F|}\pi^{2(r_1+r_2)} \det \{D(\sigma_i(\xi_j))\}_{1\le i,j\le r_2}.$$

Proof:

- "$\mathcal{B}(F)$ is $K_3(F)$"
- Borel's theorem.

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Conjecture

Let F be a number field. Let $n_+ = r_1 + r_2$, $n_- = r_2$, and $\mp = (-1)^{k-1}$. Then

- *$\mathcal{B}_k(F)$ is finitely generated of rank $n_\mp$.*
- *$\xi_1, \ldots \xi_{n_\mp}$ $\mathbb{Q}$-basis of $\mathcal{B}_k(F) \otimes \mathbb{Q}$. Then*

$$\zeta_F(k) \sim_{\mathbb{Q}^*} \sqrt{|D_F|}\pi^{kn_\pm} \det \left\{\mathcal{L}_k \left(\sigma_i \left(\xi_j\right)\right)\right\}_{1 \leq i,j \leq n_\mp} .$$

Example

$F = \mathbb{Q}(\sqrt{5})$, $r_1 = 2, r_2 = 0$.

$\left\{[1], \left[\frac{-1+\sqrt{5}}{2}\right]\right\}$ basis for $\mathcal{B}_3(F)$.

$$\begin{vmatrix} \mathcal{L}_3(1) & \mathcal{L}_3\left(\frac{-1+\sqrt{5}}{2}\right) \\ \mathcal{L}_3(1) & \mathcal{L}_3\left(\frac{-1-\sqrt{5}}{2}\right) \end{vmatrix}$$

$$= \begin{vmatrix} \zeta(3) & \frac{1}{10}\zeta(3) + \frac{25}{48}\sqrt{5}L(3,\chi_5) \\ \\ \zeta(3) & \frac{1}{10}\zeta(3) - \frac{25}{48}\sqrt{5}L(3,\chi_5) \end{vmatrix}$$

$$= -\frac{25}{24}\sqrt{5}\zeta(3)L(3,\chi_5) = -\frac{25}{24}\sqrt{5}\zeta_F(3).$$

Application

D'Andrea, L. (2007)

$$\frac{1}{(2\pi \mathrm{i})^3}\int_{\mathbb{T}^3} \log\left|z-\frac{(1-x)(1-y)}{1-xy}\right|\frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y}\frac{\mathrm{d}z}{z} = \frac{25\sqrt{5}L(3,\chi_5)}{\pi^2}$$