Introduction to modular forms

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Motivation

We borrow freely from the bibliography in these notes. This first part is mainly from Milne's notes [2].

Let X a connected Hausdorff topological space. A coordinate neighborhood of a point $P \in X$ is a pair (U, z) where $P \in U$ open, and z is a homeomorphism of U onto an open subset of $\mathbb{C}$. A complex structure on X is a compatible family of coordinate neighborhoods that cover X. A Riemann surface is a topological space together with its complex structure. Examples: any open subset of $\mathbb{C}$, the unit sphere.

Let $V \subset X$ open subset of a Riemann sphere. A function $f : V \to \mathbb{C}$ is holomorphic if for all (U, z), $f \circ z^{-1}$ is holomorphic in $z(U)$. Similarly for meromorphic functions.

Problem: study the holomorphic functions on all Riemann surfaces.

From topology there is the universal covering space $\tilde{X}$, $p : \tilde{X} \to X$ local homeomorphism. $\tilde{X}$ admits a unique complex structure for which p is a local isomorphism of Riemann surfaces. If Γ is the group of covering transformations, the $X = \Gamma \setminus \tilde{X}$.

By the Riemann mapping Theorem, $\tilde{X}$ is isomorphic to $\mathbb{C}$, $D = \{z \in \mathbb{C} \,|\, |z| < 1\}$, or the Riemann sphere.

Instead of looking at D, we look at the complex upper half plane $\mathbb{H} = \{z \in \mathbb{C} \,|\, \mathrm{Im}(z) > 0\}$, which is conformally equivalent because of the transformation $z \to \frac{z-\mathrm{i}}{z+\mathrm{i}}$.

Then we study Riemann surfaces of the form $\Gamma \setminus \mathbb{H}$, with Γ discrete group acting on $\mathbb{H}$. We need to find Γ. An obvious choice is the special linear group $SL_2(\mathbb{R})$, the action given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}.$$

Indeed,

$$\mathrm{Im}\left(\frac{az+b}{cz+d}\right) = \mathrm{Im}\left(\frac{(az+b)(c\bar{z}+d)}{|cz+d|^2}\right) = \frac{\mathrm{Im}(adz+bc\bar{z})}{|cz+d|^2} = \frac{\mathrm{Im}(z)}{|cz+d|^2}.$$

Actually there is an isomorphism

$$SL_2(\mathbb{R})/\{\pm I\} \to \mathrm{Aut}(\mathbb{H}),$$

(bi-holomorphic automorphisms of $\mathbb{H}$). An obvious discrete subgroup of $SL_2(\mathbb{R})$ is the full modular group $\Gamma = SL_2(\mathbb{Z})$. For $N \geq 0$, we have:

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,|\, a \equiv d \equiv 1 \,(\mathrm{mod} N), b \equiv c \equiv 0 \,(\mathrm{mod} N) \right\}. \quad (1)$$

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Note that the $\Gamma(N)$ are normal. In Number Theory we are interested in discrete subgroups of $SL_2(\mathbb{R})$ that contain some $\Gamma(N)$ as a finite index subgroup (congruence subgroups of level N). For example, $\Gamma_0(N)$ ($c \equiv 0\,(\mathrm{mod} N)$), $\Gamma_1(N)$ ($c \equiv 0\,(\mathrm{mod} N), a \equiv 1\,(\mathrm{mod} N)$).

Now we take $Y(N) = \Gamma(N) \setminus \mathbb{H}$ with the quotient topology. We can endow it with a (unique) structure of Riemann surface. Its compactification is denoted by $X(N)$.

The fundamental domain by the action of $SL_2(\mathbb{Z})$

Here we follow Koblitz [1]. How does $X(N)$ look like? Let us look at the case of the full modular group. The fundamental domain for the action of a group Γ in $\mathbb{H}$ is a subset F of $\mathbb{H}$ such that every point $z \in \mathbb{H}$ is Γ-equivalent to a point in F and no distinct points z_1, z_2 in the interior of F are equivalent. It turns out that

Proposition 1 *For $\Gamma = \Gamma(1)$,*

$$F = \left\{ z \in \mathbb{H} \,|\, -\frac{1}{2} \leq \mathrm{Re}(z) \leq \frac{1}{2},\, |z| \geq 1 \right\} \tag{2}$$

is a fundamental domain. PICTURE.

Idea of Proof. We use $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. First we prove that any $z \in \mathbb{H}$ is equivalent to a point in F. We do that by translating with T until the real part is less than $\frac{1}{2}$ in absoulte value, and then applying S once if necessary. Then prove that not two interior points are equivalent. This is easy but technical.

Let Γ_z be the isotropy subgroup of z, meaning, $\Gamma_z := \{\gamma \in \Gamma(1) \,|\, \gamma(z) = z\}$. Then

Proposition 2 *$z \in F$, then $\Gamma_z = \pm I$ unless*

- $\Gamma_{\mathrm{i}} = \langle S \rangle$.
- $\Gamma_\omega = \langle ST \rangle$ *for* $\omega = \frac{-1+\sqrt{3}\mathrm{i}}{2}$.
- $\Gamma_\omega = \langle TS \rangle$ *for* $\omega = \frac{1+\sqrt{3}\mathrm{i}}{2}$.

Another consequence is

Proposition 3 *The group $\Gamma(1)$ is generated by S and T.*

In order to complete $Y(1)$, we need to add the point at infinity. Only one point is enough, since Γ is transitive in $\mathbb{Q} \cup \infty$.

Modular forms

We are looking for functions that are a meromorphic on $\mathbb{H}$, invariant under $\Gamma(N)$ and meromorphic at the cusps. That means that they can be regarded as functions on $Y(N)$ and as such, they remain meromorphic when extended to $X(N)$.

In the case of the full modular group, to be invariant means that

$$f\left(\frac{az+b}{cz+d}\right) = f(z) \quad \text{for all} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}).$$

In particular taking $T \in SL_2(\mathbb{Z})$, we have that $f(z+1) = f(z)$, and thus we can write $f(z) = f^*(q)$ where $q = \mathrm{e}^{2\pi \mathrm{i} z}$. As z moves in $\mathbb{H}$, q moves in the unit punctured disk. To say that f is meromorphic at the cusps means that $f^*(q)$ is meromorphic in the whole disk,

$$f(z) = \sum_{n \geq -N_0} a_n q^n. \tag{3}$$

It is hard to construct a meromorphic function on $\mathbb{H}$ that is invariant under the action of $\Gamma(N)$. We can, instead, construct functions that transform in a "nice way" under the action of $\Gamma(N)$. The quotient of two such functions will be then a modular function (analogous to the construction of rational functions on the proyective space).

Definition 4 *A holomorphic (meromorphic) function $f(z)$ on $\mathbb{H}$ is a modular form (function) of level N and weight k if*

1.

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z) \quad \text{for all} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N). \tag{4}$$

2. $f(z)$ is holomorphic (meromorphic) at the cusps.

In particular, for the full modular group

1.

$$f(z+1) = f(z), \qquad f\left(-\frac{1}{z}\right) = (-z)^k f(z). \tag{5}$$

2.

$$f(z) = \sum_{n \geq 0} a_n q^n. \tag{6}$$

If we further have that $a_0 = 0$, the form is called a cusp-form of weight k for the full modular group.

The set of such forms is denoted by $M_k(\Gamma(N))$. The cusp forms are denoted by $S_k(\Gamma(N))$.

Notice that for k odd there are no nonzero modular functions for $\Gamma(1)$ (to see this, take $-I$).

The set of modular forms of weight k is a vector space. The product of two modular forms of weight k_1 and k_2 is a modular form of weight $k_1 + k_2$.

Examples

Eisenstein series.

Definition 5 *If k is an even integer greater than 2, define*

$$G_k(z) := \sum_{m,n}{}' \frac{1}{(mz+n)^k}, \tag{7}$$

where the summation is taken over the pairs m, n where not both are zero.

For $k \geq 4$ the sum is absolutely convergent and uniformly convergent in any compact subset of $\mathbb{H}$. Hence $G_k(z)$ is a holomorphic function. Clearly $G_k(z+1) = G_k(z)$, and the Fourier expansion has no negative terms since

$$\lim_{z \to \mathrm{i}\infty} \sum_{m,n}{}' \frac{1}{(mz+n)^k} = 2 \sum_{n \neq 0} \frac{1}{n^k} = 2\zeta(k).$$

Also

$$G_k\left(-\frac{1}{z}\right) = \sum_{m,n}{}' \frac{z^k}{(-m+nz)^k} = z^k G_k(z).$$

Then $G_k(z) \in M_k(\Gamma(1))$.

Proposition 6 *The q-expansion of $G_k(z)$ is given by*

$$G_k(z) = 2\zeta(k) \left(1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \right), \tag{8}$$

where

$$\sigma_{k-1}(n) := \sum_{d|n} d^{k-1},$$

and B_k is the kth Bernoulli number given by

$$\frac{x}{\mathrm{e}^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!}.$$

For the proof use that

$$\pi\mathrm{i} + \frac{2\pi\mathrm{i}}{\mathrm{e}^{2\pi\mathrm{i}a} - 1} = \pi\cot(\pi a) = \frac{1}{a} + \sum_{n=1}^{\infty} \left(\frac{1}{a+n} + \frac{1}{a-n}\right) \quad a \in \mathbb{H},$$

and differentiate many times. (This also proves that $\zeta(2k) = \frac{2^{2k-1}}{(2k)!} B_{2k} \pi^{2k}$).

Write $E_k(z) = \frac{G_k(z)}{2\zeta(k)}$. The first few examples are:

$$E_4(z) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n,$$

$$E_6(z) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n.$$

Another example (we follow [2] again): the (finite-index) quotients of $\mathbb{C}$ are given by lattices

$$\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2,$$

where the ω_i are complex numbers whose quotient is not real, (the quotient may be taken in such a way that $\mathrm{Im}\left(\frac{\omega_2}{\omega_1}\right) > 0$. Then $\mathbb{C}/\Lambda$ is a torus and it can be given a unique complex structure. A meromorphic function must satisfy $f(z+\lambda) = f(z)$ for every $\lambda \in \Lambda$. Consider

$$\mathcal{P}(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda,\, \lambda \neq 0} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\right).$$

This is a meromorphic function, invariant under Λ, and

$$[z] \to (\mathcal{P}(z) : \mathcal{P}'(z) : 1)$$

defines an isomorphism of the Riemann surface $\mathbb{C}/\Lambda$ onto the Riemann surface $E(\mathbb{C})$ where E is the elliptic curve

$$Y^2 Z = 4X^3 - g_2 X Z^2 - g_3 Z^3,$$

for certain g_2, g_3. It turns out that

$$E(\Lambda) \cong E(\Lambda') \Leftrightarrow \Lambda' = c\Lambda \quad c \in \mathbb{C}^*.$$

Then we can assume $\Lambda = \mathbb{Z} \oplus \mathbb{Z}\tau$, with $\tau \in \mathbb{H}$. Then

$$g_2 = 60 G_4(\tau), \qquad g_3 = 140 G_6(\tau).$$

Indeed, this actually defines an isomorphism

$$Y(1) \to \{\text{elliptic curves over } \mathbb{C}\}/\cong$$

$$\tau \to E(\tau)$$

Now consider $\Delta = g_2^3 - 27 g_3^2$. It is the discriminant of the curve and is different from zero. It is a modular form of weight 12 given by

$$(2\pi)^{-12}\Delta = q \prod_{n=1}^{\infty} (1 - q^n)^{24} := \sum_{n=1}^{\infty} \tau(n) q^n.$$

It was studied by Ramanujan and it has many properties, like $\tau(mn) = \tau(m)\tau(n)$ when m, n are coprime.

The function

$$j(\tau) := 1728 g_2(\tau)^3/\Delta(\tau)$$

is a modular function of weight 0 for $\Gamma(1)$ and defines an isomorphism

$$j : Y(1) \to \mathbb{C}.$$

$$j(\tau) = \frac{1}{q} + 744 + \sum_{n=1}^{\infty} c(n) q^n$$

Arithmetic facts about elliptic curves translate in this way into arithmetic facts of special values of modular forms.

Proposition 7 *Let $f(z)$ be a nonzero modular function of weight k for $\Gamma(1)$. For $P \in \mathbb{H}$, let $v_P(f)$ denote the order of zero (taken with negative sign for poles) of $f(z)$ at the point P. Let $v_\infty(f)$ denote the order at infinity (the index of the first non vanishing term in the Fourier expansion). Then*

$$v_\infty(f) + \frac{1}{2} v_{\mathrm{i}}(f) + \frac{1}{3} v_\omega(f) + \sum_{P \in \Gamma(1)\backslash\mathbb{H}, P \neq \mathrm{i}, \omega} v_P(f) = \frac{k}{12}$$

Idea of the proof: count the number of zeros and poles in $\Gamma(1) \setminus \mathbb{H}$ by integrating the logarithmic derivative $\frac{f'}{f}$ around the boundary of the fundamental domain and playing with the Residue Theorem.

Corollary 8 *Let k be an even integer.*

- *The only modular forms of weight* 0 *are the constants.*
- $M_k(\Gamma(1)) = 0$ *if k is negative or $k = 2$.*
- $M_k(\Gamma(1))$ *is one-dimensional, generated by E_k for $k = 4, 6, 8, 10$ or* 14.
- $S_k(\Gamma(1)) = 0$ *if $k < 12$ or $k = 14$. $S_{12}(\Gamma(1)) = \mathbb{C}\Delta$. For $k > 14$, $S_k(\Gamma(1)) = \Delta M_{k-12}(\Gamma(1))$.*
- $M_k(\Gamma(1)) = S_k(\Gamma(1)) \oplus \mathbb{C}E_k$ *for $k > 2$.*
- *For $k \geq 0$,*

$$\dim M_k(\Gamma(1)) = \begin{cases} \left[\frac{k}{12}\right] & k \equiv 1 \,(\mathrm{mod} 12) \\ \left[\frac{k}{12}\right] + 1 & k \not\equiv 1 \,(\mathrm{mod} 12) \end{cases}$$

Corollary 9 *Any $f \in M_k(\Gamma(1))$ can be can be written as*

$$f(z) = \sum_{4i+6j=k} c_{i,j} E_4(z)^i E_6(z)^j \tag{9}$$

Proposition 10 *The modular functions of weight zero for Γ are the rational functions for j.*

References

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