APPLICATIONS OF MULTIZETA VALUES TO MAHLER MEASURE

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Abstract. These notes correspond to a mini-course taught by the author during the program "PIMS-SFU undergraduate summer school on multiple zeta values: combinatorics, number theory and quantum field theory". Please send any comments or corrections to the author at mlalin@dms.umontreal.ca.

1. Primes, Mahler Measure, and Lehmer's question

We start our study by discussing prime numbers. First consider the sequence of numbers $M_n = 2^n - 1$ for n natural. We may ask, when is M_n prime? We can write

$$(2^{rs} - 1) = (2^r - 1)(2^{(s-1)r} + 2^{(s-2)r} + \cdots + 2^r + 1).$$

This implies that $M_r \mid M_{rs}$ and we need n prime in order for M_n to be prime. Notice that $M_2 = 3, M_3 = 7, M_5 = 31, M_7 - 127$, but $M_{11} = 2^{11} - 1 = 2047 = 23 \times 89$. Therefore the converse is not true.

The primes of the form M_p are called Mersenne primes. It is unknown if there are infinitely many of such primes, or if there are infinitely many M_p composite with p prime. The largest Mersenne prime known to date is with $p = 57, 885, 161$ (it has $17, 425, 170$ digits). The search for large Mersenne primes is being carried by the "Great Internet Mersenne Prime Search": http://www.mersenne.org.

***Exercise* 1.** Let a, p be natural numbers such that $a^p - 1$ is prime, then show that either $a = 2$ or $p = 1$

***Exercise* 2.** Let p be an odd prime. Show that every prime q that divides $2^p - 1$ must be of the form $q = 2pk + 1$ with k integer.

Looking for large primes, Pierce [Pi17] proposed the following construction in 1917. Consider $P \in \mathbb{Z}[x]$ monic, and write

$$P(x) = \prod_i (x - \alpha_i)$$

then, we look at

$$\Delta_n = \prod_i (\alpha_i^n - 1).$$

The α_i are algebraic integers. By applying Galois theory, it is easy to see that $\Delta_n \in \mathbb{Z}$. Note that if $P = x - 2$, we get the sequence $\Delta_n = 2^n - 1$, the Mersenne numbers. The idea is to look for primes among the factors of Δ_n. The prime divisors of such integers must satify some congruence conditions that are quite restrictive, hence they are easier to factorize than a randomly given number.

***Exercise* 3.** Prove that Δ_n is a divisibility sequence, namely, if $n \mid m$, then $\Delta_n \mid \Delta_m$.

Then we may look at the numbers

$$\frac{\Delta_p}{\Delta_1}, \qquad p \text{ prime.}$$

Pierce and Lehmer observed that the only possible factors of Δ_n are given by prime powers p^e of the form $nk+1$ for some integer k and $1 \le e \le \deg(P)$. It is then natural to look for P that generate sequences that grow slowly so that they have a small chance of having factors. Lehmer [Le33] studied $\frac{|\Delta_{n+1}|}{|\Delta_n|}$, observed that

$$\lim_{n\to\infty} \frac{|\alpha^{n+1} - 1|}{|\alpha^n - 1|} = \begin{cases} |\alpha| & \text{if } |\alpha| > 1, \\ 1 & \text{if } |\alpha| < 1, \end{cases}$$

and suggested the following definition:

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Definition 1.1. Given $P \in \mathbb{C}[x]$, such that

$$P(x) = a \prod_i (x - \alpha_i)$$

define the (Mahler) measure * of P as

$$M(P) = |a| \prod_i \max\{1, |\alpha_i|\}.$$

The logarithmic Mahler measure is defined as

$$\mathrm{m}(P) = \log M(P) = \log |a| + \sum_i \log^+ |\alpha_i|,$$

where $\log^+ |\alpha| = \log \max\{1, |\alpha|\}$.

***Exercise* 4.** Prove that for any $n \in \mathbb{Z}$,

$$\mathrm{m}(P(x)) = \mathrm{m}(P(x^n)).$$

***Exercise* 5.** Prove that if $P(x) = a_d x^d + \cdots + a_0$, then $|a_i| \leq \binom{d}{i} M(P)$.

As $M(P)$ measures the growth of the sequence $\frac{|\Delta_{n+1}|}{|\Delta_n|}$, it is natural to ask about the sequences that do not grow: When does $M(P) = 1$ for $P \in \mathbb{Z}[x]$? We have

Lemma 1.2. *(Kronecker,* [Kr57]*) Let $P = \prod_i (x - \alpha_i) \in \mathbb{Z}[x]$, if $|\alpha_i| \leq 1$, then the α_i are zero or roots of the unity.*

Proof. Consider the polynomial

$$P_n(x) = \prod_{i=1}^{d} (x - \alpha_i^n).$$

The coefficients of $P_n(x)$ are symmetric functions in the algebraic integers α_i^n, so they are elements of $\mathbb{Z}$ (all the conjugates of each α_i are present as roots of $P(x)$, since the coefficients are rational). Each of the coefficients is uniformly bounded as n varies, because $|\alpha_j| \leq 1$ and the set $\{P_n\}_{n \in \mathbb{N}}$ must be finite. In other words, there are $n_1 \neq n_2$ for which

$$P_{n_1} = P_{n_2}.$$

That means,

$$\{\alpha_1^{n_1}, \ldots, \alpha_d^{n_1}\} = \{\alpha_1^{n_2}, \ldots, \alpha_d^{n_2}\}.$$

Thus, there is a permutation $\sigma \in \mathbb{S}_d$ such that

$$\alpha_i^{n_1} = \alpha_{\sigma(i)}^{n_2}.$$

If σ has order k, we get,

$$\alpha_i^{n_1^k} = \alpha_i^{n_2^k},$$

and α_i is a root of

$$x^{n_1^k}(x^{n_2^k - n_1^k} - 1) = 0.$$

This shows that each α_i is either zero or a root of unity. ☺

*The name Mahler came about 30 years later after the person who successfully extended this definition to the several-variable case.

***Exercise* 6.** (a) Give examples to show that in general a polynomial in $\mathbb{Z}[x]$ may have zeros of absolute value one that are not roots of the unity.

(b) Show that a monic example of (a) can only occur in degree at least 4.

By Kronecker's Lemma, $P \in \mathbb{Z}[x]$, $P \neq 0$, then $M(P) = 1$ if and only if P is the product of powers of x and cyclotomic polynomials. This statement characterizes integral polynomials whose Mahler measure is 1.

Lehmer found the example

$$\mathrm{m}(x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1) = \log(1.176280818\dots) = 0.162357612\dots$$

and asked the following (Lehmer's question, 1933):

Is there a constant $C > 1$ such that for every polynomial $P \in \mathbb{Z}[x]$ with $M(P) > 1$, then $M(P) \geq C$?

Lehmer's question remains open nowadays. His 10-degree polynomial remains the best possible result.

***Exercise* 7.** With the help of a computer find the Mahler measures of the following polynomials

(a) $x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$,

(b) $x^{10} - x^6 + x^5 - x^4 + 1$,

(c) $x^{14} + x^{11} - x^{10} - x^7 - x^4 + x^3 + 1$,

(d) $x^{14} - x^{12} + x^7 - x^2 + 1$.

Examples of polynomials with small Mahler measure may be found with the search engine from Mossinghoff's website [M]: `http://www.cecm.sfu.ca/~mjm/Lehmer/search/`.

***Exercise* 8.** With the help of a computer investigate the sequences Δ_n for $x^{10}+x^9-x^7-x^6-x^5-x^4-x^3+x+1$ and for $x^3 - x - 1$. Challenge: Find $\sqrt{\Delta_{4951}}$ for the first polynomial and Δ_{23311} for the second polynomial and check that they are prime numbers. Hint: it may be necessary to verify that the Δ_n satisfy a recurrence sequence as proved in [Le33], section 8.

The following definiton will be used often in the notes.

Definition 1.3. Let $P(x) \in \mathbb{C}[x]$ be a nonzero polynomial of degree d. We set

$$P^*(x) := x^d \overline{P}(x^{-1})$$

the polynomial with the same coefficients as P, but in reverse order and conjugated. We say that P is reciprocal if $P = \pm P^*$. Otherwise, we say that P is non-reciprocal.

We list here some important results in the direction of solving Lehmer's question.

Theorem 1.4. *(Breusch* [Br51]*, Smyth* [Sm71]*) If $P \in \mathbb{Z}[x]$ is monic, irreducible, non-reciprocal, then*

$$M(P) \geq M(x^3 - x - 1) = \theta = 1.324717\dots$$

Corollary 1.5. *If $P \in \mathbb{Z}[x]$ is monic, irreducible, and of odd degree, then*

$$M(P) \geq \theta.$$

The most general result with a bound involving the degree is given by

Theorem 1.6. *(Dobrowolski* [Do79]*) If $P \in \mathbb{Z}[x]$ is monic, irreducible and noncyclotomic of degree d, then*

$$M(P) \geq 1 + c\left(\frac{\log\log d}{\log d}\right)^3$$

where c is an absolute positive constant.

Theorem 1.7. *(Schinzel) If $P \in \mathbb{Z}[x]$ is monic of degree d having all real roots and satisfies $P(1)P(-1) \neq 0$ and $|P(0)| = 1$, then*

$$M(P) \geq \left(\frac{1+\sqrt{5}}{2}\right)^{\frac{d}{2}}$$

with equality iff P is a power of $x^2 - x - 1$.

For the proof, we need the following

***Exercise* 9.** For any $d \geq 1$ and $y_1, \dots, y_d > 1$ be real numbers, prove that

$$(y_1 - 1) \cdots (y_d - 1) \leq ((y_1 \cdots y_d)^{1/d} - 1)^d.$$

Hint: This is a special case of Mahler's inequality. Apply the arithmetic–geometric mean inequality to $\left\{\frac{y_1-1}{y_1}, \dots, \frac{y_d-1}{y_d}\right\}$ and to $\left\{\frac{1}{y_1}, \dots, \frac{1}{y_d}\right\}$.

Proof. (Theorem 1.7) Consider

$$E = \prod_{i=1}^{d} |\alpha_i^2 - 1|.$$

Remark that $E \geq 1$ since P is monic and $\alpha_i \neq \pm 1$. Note that

$$M(P) = \prod_{|\alpha_i|>1} |\alpha_i| = \frac{1}{\prod_{|\alpha_i|<1} |\alpha_i|}.$$

Thus, we may rewrite

$$E = \prod_{|\alpha_i|<1} |\alpha_i^2 - 1| \prod_{|\alpha_i|>1} |\alpha_i^2 - 1| = \frac{1}{M(P)^2} \prod_{|\alpha_i|<1} |\alpha_i^{-2} - 1| \prod_{|\alpha_i|>1} |\alpha_i^2 - 1|.$$

We apply Exercise 9 in order to obtain

$$\begin{aligned} E &\leq \frac{1}{M(P)^2} \left(\left(\prod_{|\alpha_i|<1} \alpha_i^{-2} \prod_{|\alpha_i|>1} \alpha_i^2 \right)^{1/d} - 1 \right)^d \\ &= \frac{1}{M(P)^2} \left(M(P)^{4/d} - 1 \right)^d \\ &= (M(P)^{2/d} - M(P)^{-2/d})^d. \end{aligned}$$

Since $E \geq 1$,

$$M(P)^{2/d} - M(P)^{-2/d} \geq 1. \tag{1.1}$$

Since $M(P) > 1$, this implies that $M(P)^{2/d} \geq \frac{1+\sqrt{5}}{2}$ and we obtain the desired result. ☺

***Exercise* 10.** Prove the last assertion of Theorem 1.7.

***Exercise* 11.** What happens if we relax the condition of $|P(0)| = 1$?

Theorem 1.8. *(Bombieri, Vaaler* [BV87]*) Let $P \in \mathbb{Z}[x]$ with $M(P) < 2$, then P divides a polynomial $Q \in \mathbb{Z}[x]$ whose coefficients belong to $\{-1, 0, 1\}$.*

The last result and Theorem 1.4 suggest that if we want to beat Lehmer's 10-degree polynomial, one should search for reciprocal polynomials having coefficients in $\{-1, 0, 1\}$.

The most general result in terms of families is given by

Theorem 1.9. *(Borwein, Dobrowolski, Mossinghoff* [BDM07]*) Let $\mathcal{D}_m$ denote the set of polynomials whose coefficients are all congruent to 1 modulo m. Si $f \in \mathcal{D}_m$ has degree d and no cyclotomic factors, then*

$$\mathrm{m}(f) \geq c_m \left(1 - \frac{1}{d+1}\right),$$

where $c_2 = (\log 5)/4$ and $c_m = \log(\sqrt{m^2+1}/2)$ for $m > 2$.

As a final comment, we remark that Mahler measure is related to classical heights. Generally speaking, a height is a function that measures the size of a mathematical object. For example, the absolute value $|\cdot|$ measures the size of complex numbers. Another less immediate example is given by the canonical height defined in $\mathbb{Q}$ as follows. If $\frac{a}{b}$ is a rational number written in lowest terms (in other words, $(a, b) = 1$), then the height of $\frac{a}{b}$ is defined as $\max\{|a|, |b|\}$. This definition is what allows us to disntiguish between the number 2 and the number $1.9999999999 = \frac{19{,}999{,}999{,}999}{10{,}000{,}000{,}000}$: while the first has height 2, the second one has height 19,999,999,999.

For a polynomial $P(x) = a_d x^d + \cdots + a_1 x + a_0 \in \mathbb{C}[x]$ define

$$H(P) = \max_{0 \le i \le d} \{|a_i|\}, \quad L(P) = \sum_{i=0}^{d} |a_i|,$$

the height and the length of P.

***Exercise* 12.** Prove the following inequalities:
(a) $L(P) \le 2^d M(P)$.
(b) $H(P) \le 2^{1-d} M(P)$.

Mahler [Ma62] considered this construction because he was looking for inequalities of the classical polynomial heights (such as $L(P)$ or $H(P)$) between the height of a product of polynomials and the heights of the factors. These kinds of inequalities are useful in transcendence theory. $M(P)$ is multiplicative (that is, $M(PQ) = M(P)M(Q)$) and comparable to the typical heights, and that makes it possible to deduce such inequalities. While investigating this in several variables, he discovered the generalization that we explore in the next section.

2. Mahler Measure in several variables

We will be concerned mostly with the Mahler measure of multivariable polynomials.

Definition 2.1. For $P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]^\times$, the logarithmic Mahler measure is defined by

$$\begin{aligned} \mathrm{m}(P) &:= \int_0^1 \cdots \int_0^1 \log |P(e^{2\pi i\theta_1}, \dots, e^{2\pi i\theta_n})| d\theta_1 \dots d\theta_n \\ &= \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log |P(x_1, \dots, x_n)| \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}. \end{aligned}$$

The Mahler measure is defined by

$$M(P) := e^{\mathrm{m}(P)}.$$

It is possible to prove that this integral is not singular and that $\mathrm{m}(P)$ always exists. This definition appeared for the first time in the work of Mahler [Ma62].

The relationship between Definitions 2.1 and 1.1 is given by the following.

Theorem 2.2. *(Jensen's formula) Let $\alpha \in \mathbb{C}$. Then*

$$\int_0^1 \log |e^{2\pi i\theta} - \alpha| d\theta = \log^+ |\alpha|$$

where $\log^+ x = \log \max\{1, x\}$ *for* $x \in \mathbb{R}_{\ge 0}$.

Proof. First assume that $|\alpha| < 1$. We have that

$$\begin{aligned} \int_0^1 \log |e^{2\pi i\theta} - \alpha| d\theta &= \int_0^1 \log |1 - \alpha e^{-2\pi i\theta}| d\theta \\ &= \int_0^1 \log |1 - \alpha e^{2\pi i\tau}| d\tau \end{aligned}$$

where we have done $\tau = -\theta$. We have

$$\begin{aligned}\int_0^1 \log|1-\alpha e^{2\pi i\tau}|d\tau &= \mathrm{Re}\left(\int_0^1 \log(1-\alpha e^{2\pi i\tau})d\tau\right)\\ &= \mathrm{Re}\int_0^1 \left(-\sum_{n=1}^{\infty}\frac{\alpha^n}{n}e^{2\pi i\tau n}\right)d\tau\\ &= \mathrm{Re}\left(-\sum_{n=1}^{\infty}\frac{\alpha^n}{n}\int_0^1 e^{2\pi i\tau n}d\tau\right)\\ &= 0.\end{aligned}$$

We have exchanged the integral and the infinite sum, but this is justified since the sum is absolutely convergent.

Now assume that $|\alpha| > 1$. Then, by the previous argument,

$$\int_0^1 \log|e^{2\pi i\theta}-\alpha|d\theta = \log|\alpha| + \int_0^1 \log|1-\alpha^{-1}e^{2\pi i\theta}|d\theta = \log|\alpha|.$$

Finally, we are left with the hardest case, $|\alpha| = 1$. After multiplying by α^{-1}, we may assume that $\alpha = 1$. Writing $|1-e^{2\pi i\theta}| = 2\sin(\pi\theta)$ for $0 \leq \theta \leq 1$, we have

$$\int_0^1 \log|1-e^{2\pi i\theta}|d\theta = \int_0^1 \log\sin(\pi\theta)d\theta + \log 2. \tag{2.1}$$

Let $I = \int_0^1 \log\sin(\pi\theta)d\theta$. The integral exists since $\sin(\pi\theta) \sim \pi\theta$ for small θ.

We write $\sin(\pi\theta) = 2\sin\left(\frac{\pi\theta}{2}\right)\cos\left(\frac{\pi\theta}{2}\right)$. Thus,

$$I = \log 2 + \int_0^1 \log\sin\left(\frac{\pi\theta}{2}\right)d\theta + \int_0^1 \log\cos\left(\frac{\pi\theta}{2}\right)d\theta.$$

By making the change $\tau = \theta/2$ in the first integral and $\tau = 1/2-\theta/2$ in the second one, we obtain,

$$I = \log 2 + 4\int_0^{1/2} \log\sin(\pi\tau)d\tau = \log 2 + 2\int_0^1 \log\sin(\pi\tau)d\tau = \log 2 + 2I.$$

From this, $I = -\log 2$ and we obtain the desired result by combining with equation (2.1). ☺

The multivariable Mahler measure is still multiplicative, meaning that for $P, Q \in \mathbb{C}[x_1, \ldots, x_n]$, we have

$$\mathrm{m}(P\cdot Q) = \mathrm{m}(P) + \mathrm{m}(Q).$$

Proposition 2.3. *Let $P \in \mathbb{C}[x_1, \ldots, x_n]$ such that $a_{i_1,\ldots,i_n}$ is the coefficient of $x_1^{i_1}\ldots x_n^{i_n}$ and P has degree d_i in x_i. Then*

(a)

$$|\,a_{i_1,\ldots,i_n}| \leq \binom{d_1}{i_1}\cdots\binom{d_n}{i_n}M(P)$$

(b)

$$M(P) \leq L(P) \leq 2^{d_1+\cdots+d_n}M(P)$$

(c)

$$((d_1+1)\cdots(d_n+1))^{-1/2}M(P) \leq H(P) \leq 2^{d_1+\cdots+d_n-n}M(P)$$

where

$$H(P) = \max_{0\leq i_j\leq d_j, j=1,\ldots,n}\{|a_{i_1,\ldots,i_n}|\}, \quad L(P) = \sum_{j=1}^{n}\sum_{i_j=0}^{d_j}|a_{i_1,\ldots,i_n}|.$$

Proof. [(c), lower bound] By an inequality of Hardy–Littlewood–Pólya,

$$M(P) \leq \left(\int_0^1 \cdots \int_0^1 |P(e^{2\pi i \theta_1}, \ldots, e^{2\pi i \theta_n})|^2 d\theta_1 \ldots d\theta_n \right)^{1/2}.$$

Parseval's formula implies

$$\begin{aligned} \int_0^1 \cdots \int_0^1 |P(e^{2\pi i \theta_1}, \ldots, e^{2\pi i \theta_n})|^2 d\theta_1 \ldots d\theta_n &= \sum_{i_j} |a_{i_1,\ldots,i_n}|^2 \\ &\leq (d_1+1)\ldots(d_n+1) H(P)^2. \end{aligned}$$

☺

***Exercise* 13.** Prove the rest of Proposition 2.3

***Exercise* 14.** If $P \in \mathbb{C}[x_1, x_2]$ has a constant coefficient a that in absolute value exceeds the sum of the absolute values of all the other coefficients, prove that $\mathrm{m}(P) = \log |a|$.

***Exercise* 15.** Let

$$P(\mathbf{x}) = \sum_{\mathbf{m}} c_{\mathbf{m}} \mathbf{x}^{\mathbf{m}} \in \mathbb{C}[x_1, \ldots, x_n],$$

where $\mathbf{x}^{\mathbf{m}} = x_1^{m_1} \cdots x_n^{m_n}$. Let A be an $n \times n$ integer matrix with non-zero determinant, and define

$$P^{(A)}(\mathbf{x}) := \sum_{\mathbf{m}} c_{\mathbf{m}} \mathbf{x}^{A\mathbf{m}}.$$

Prove that

$$\mathrm{m}(P) = \mathrm{m}\left(P^{(A)}\right).$$

It is also true that $\mathrm{m}(P) \geq 0$ if P has integral coefficients.

In addition, an analogous of Kronecker's lemma is true.

Theorem 2.4. *(Smyth* [Sm82]*) For any primitive polynomial (i.e., the coefficients have no nontrivial common factor) $P \in \mathbb{Z}[x_1^{\pm}, \ldots, x_n^{\pm}]$, $\mathrm{m}(P)$ is zero if and only if P is a monomial times a product of cyclotomic polynomials evaluated on monomials.*

Let us also mention the following result:

Theorem 2.5. *(Boyd* [Bo81]*, Lawton* [La83]*) For $P \in \mathbb{C}[x_1, \ldots, x_n]$*

$$\lim_{k_2 \to \infty} \cdots \lim_{k_n \to \infty} \mathrm{m}(P(x, x^{k_2}, \ldots, x^{k_n})) = \mathrm{m}(P(x_1, \ldots, x_n)).$$

It should be noted that the limit has to be taken independently for each variable. (Writing this properly would take half a page.)

***Exercise* 16.** Explore the limit of Theorem 2.5 in the case of $1+x+y$. Namely, compute several values of $\mathrm{m}(1+x+x^n)$ and compare them with the value of $\mathrm{m}(1+x+y) = 0.323065947\ldots$.

Because of the above theorem, Lehmer's question in the several-variable case reduces to the one-variable case. In addition, this theorem shows us that we are working with the "right" generalization of the original definition for one-variable polynomials.

The formula for the one-variable case tells us some information about the nature of the values that Mahler measure can reach. For instance, the Mahler measure of a polynomial in one variable with integer coefficients must be an algebraic number.

It is natural, then, to wonder what happens with the several-variable case. Is there any simple formula, besides the integral? (Un)fortunately,† this case is much more complicated and we only have some particular examples. On the other hand, the values are very interesting.

†If it were that easy this area would not be so interesting!

3. Examples

We show some examples of formulas for Mahler measures of multivariable polynomials.

- Smyth [Sm82]

$$\mathrm{m}(x+y+1) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2) = L'(\chi_{-3}, -1), \tag{3.1}$$

where

$$L(\chi_{-3}, s) = \sum_{n=1}^{\infty} \frac{\chi_{-3}(n)}{n^s} \quad \text{and} \quad \chi_{-3}(n) = \begin{cases} 1 & \text{if} \quad n \equiv 1 \mod 3 \\ -1 & \text{if} \quad n \equiv -1 \mod 3 \\ 0 & \text{if} \quad n \equiv 0 \mod 3 \end{cases}$$

is a Dirichlet L-function.

- Smyth [Bo81]

$$\mathrm{m}(x+y+z+1) = \frac{7}{2\pi^2}\zeta(3), \tag{3.2}$$

where

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

is the Riemann zeta function.

- Boyd & L. (2005)

$$\mathrm{m}(x^2+1+(x+1)y+(x-1)z) = \frac{1}{\pi} L(\chi_{-4}, 2) + \frac{21}{8\pi^2}\zeta(3).$$

- L. [La03]

$$\mathrm{m}\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)(1+y)z\right) = \frac{93}{\pi^4}\zeta(5).$$

- In fact, there are known formulas ([La06]) for

$$\mathrm{m}\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_n}{1+x_n}\right)(1+y)z\right).$$

- Rogers & Zudilim [RZ11]

$$\mathrm{m}\left(x+\frac{1}{x}+y+\frac{1}{y}+1\right) = \frac{15}{4\pi^2} L(E_{15}, 2) = L'(E_{15}, 0)$$

where E_{15} is an elliptic curve (of conductor 15) that happens to be the algebraic closure of the zero set of the polynomial.

Roughly speaking, an ellitic curve is the zero set of a polynomial of the form

$$Y^2 = X^3 + AX + B$$

where $4A^3 + 27B^2 \neq 0$.

The polynomial $x + \frac{1}{x} + y + \frac{1}{y} + k$ generally corresponds to an elliptic curve under the following transformation

$$x = \frac{kX - 2Y}{2X(X-1)} \qquad y = \frac{kX + 2Y}{2X(X-1)}.$$

$$Y^2 = X\left(X^2 + \left(\frac{k^2}{4} - 2\right)X + 1\right).$$

(It is then easy to eliminate the term of X^2 by completing the cube.)

The L-function of an elliptic curve is a function similar to the Riemann zeta function with coefficients that encode the number of points of the curve over finite fields.

- Boyd (2005)

$$\mathrm{m}(z + (x+1)(y+1)) \stackrel{?}{=} 2L'(E_{15}, -1).$$

The question mark stands for an identity that has been verified up to 20 decimal places, but for which no proof is known.

- Examples with $K3$ surfaces, mostly due to Bertin. These includes polynomials in the family $x+\frac{1}{x}+y+\frac{1}{y}+z+\frac{1}{z}+k$.

How do we get such formulas? Some of them are very difficult to prove. To be concrete, we are going to show the proof of the first example by Smyth (from [Bo81]):

Proof. (Equation (3.1)) By Jensen's formula,

$$\mathrm{m}(1+x+y)=\frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\log|1+e^{it}+e^{is}|dtds=\frac{1}{2\pi}\int_{-\pi}^{\pi}\log\max\{|1+e^{it}|,1\}dt=\frac{1}{2\pi}\int_{-2\pi/3}^{2\pi/3}\log|1+e^{it}|dt.$$

Now we write

$$\log|1+e^{it}|=\mathrm{Re}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}e^{int}.$$

The series does not converge absolutely but it converges uniformly in $t\in[-2\pi/3,2\pi/3]$, since we are far from the singularity at $t=\pm\pi$. It follows from

$$\int_{-2\pi/3}^{2\pi/3}e^{int}dt=\frac{2}{n}\sin\frac{2n\pi}{3}=\frac{\sqrt{3}}{n}\chi_{-3}(n)$$

that

$$\mathrm{m}(1+x+y)=\frac{\sqrt{3}}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\chi_{-3}(n)}{n^2}=\frac{\sqrt{3}}{2\pi}\left(\sum_{n=1}^{\infty}\frac{\chi_{-3}(n)}{n^2}-2\sum_{n=1}^{\infty}\frac{\chi_{-3}(2n)}{(2n)^2}\right) \tag{3.3}$$

and use that $\chi_{-3}(2n)=\chi_{-3}(2)\chi_{-3}(n)=-\chi_{-3}(n)$ to obtain the initial formula. ☺

***Exercise* 17.** Prove Smyth's formula (3.2). Hint:

$$\mathrm{m}(1+x+y+z)=\mathrm{m}(1+x+z(1+y))=\int_0^1\int_0^1\log\max\left\{\left|1+e^{2\pi i\theta_1}\right|,\left|1+e^{2\pi i\theta_2}\right|\right\}d\theta_1d\theta_2.$$

Some of the formulas explored in this section can be proved and better understood by using polylogarithms.

4. Polylogarithms

Many examples should be understood in the context of polylogarithms.

Definition 4.1. The *kth polylogarithm* is the function defined by the power series

$$\mathrm{Li}_k(x):=\sum_{n=1}^{\infty}\frac{x^n}{n^k}\qquad x\in\mathbb{C},\quad |x|<1.$$

This function can be continued analytically to $\mathbb{C}\setminus[1,\infty)$.

In order to avoid discontinuities, and to extend polylogarithms to the whole complex plane, several modifications have been proposed. Zagier [Z91] considers the following version:

$$\mathcal{L}_k(x):=\mathrm{Re}_k\left(\sum_{j=0}^{k}\frac{2^jB_j}{j!}(\log|x|)^j\mathrm{Li}_{k-j}(x)\right),$$

where B_j is the jth Bernoulli number, $\mathrm{Li}_0(x)\equiv-\frac{1}{2}$ and Re_k denotes Re or Im depending on whether k is odd or even.

This function is one-valued, real analytic in $\mathbb{P}^1(\mathbb{C})\setminus\{0,1,\infty\}$ and continuous in $\mathbb{P}^1(\mathbb{C})$. Moreover, $\mathcal{L}_k$ satisfies very clean functional equations. The simplest ones are

$$\mathcal{L}_k\left(\frac{1}{x}\right)=(-1)^{k-1}\mathcal{L}_k(x)\qquad\mathcal{L}_k(\bar{x})=(-1)^{k-1}\mathcal{L}_k(x).$$

There are also lots of functional equations which depend on the index k. For instance, for $k=2$, we have the Bloch–Wigner dilogarithm,

$$D(x):=\mathrm{Im}\,(\mathrm{Li}_2(x))+\arg(1-x)\log|x|$$

which satisfies the well-known five-term relation

$$D(x) + D(1-xy) + D(y) + D\left(\frac{1-y}{1-xy}\right) + D\left(\frac{1-x}{1-xy}\right) = 0.$$

The dilogarithm can be also recovered in terms of an integral.

$$-2\int_0^\theta \log|2\sin t|dt = D(e^{2i\theta}) = \sum_{n=1}^\infty \frac{\sin(2n\theta)}{n^2}$$

More generally, recall the definition for polylogarithms.

Definition 4.2. Multiple polylogarithms are defined as the power series

$$\mathrm{Li}_{n_1,\ldots,n_m}(x_1,\ldots,x_m) := \sum_{0<k_1<k_2<\cdots<k_m} \frac{x_1^{k_1}x_2^{k_2}\ldots x_m^{k_m}}{k_1^{n_1}k_2^{n_2}\ldots k_m^{n_m}}$$

which are convergent for $|x_i| < 1$. The weight of a polylogarithm function is the number $w = n_1 + \cdots + n_m$.

When $n_m > 1$ the above series converges for $|x_i| \leq 1$. We can then find multizeta values by setting $x_i = 1$:

$$\mathrm{Li}_{n_1,\ldots,n_m}(1,\ldots,1) = \zeta(n_1,\ldots,n_m).$$

***Exercise* 18.** (a) Express $\mathrm{Li}_n(-1)$ in terms of zeta functions.
(b) Express $\mathrm{Li}_n(e^{2\pi i/3}) - \mathrm{Li}_n(e^{-2\pi i/3})$ in terms of $L(\chi_{-3}, n)$.
(c) What is $\mathrm{Li}_n(i) - \mathrm{Li}_n(-i)$?

Definition 4.3. Hyperlogarithms are defined as the iterated integrals

$$\mathrm{I}_{n_1,\ldots,n_m}(a_1 : \cdots : a_m : a_{m+1}) :=$$

$$\int_0^{a_{m+1}} \underbrace{\frac{dt}{t-a_1}\circ\frac{dt}{t}\circ\cdots\circ\frac{dt}{t}}_{n_1}\circ\underbrace{\frac{dt}{t-a_2}\circ\frac{dt}{t}\circ\cdots\circ\frac{dt}{t}}_{n_2}\circ\cdots\circ\underbrace{\frac{dt}{t-a_m}\circ\frac{dt}{t}\circ\cdots\circ\frac{dt}{t}}_{n_m}$$

where n_i are integers, a_i are complex numbers, and

$$\int_0^{b_{k+1}} \frac{dt}{t-b_1}\circ\cdots\circ\frac{dt}{t-b_k} = \int_{0\leq t_1\leq\cdots\leq t_k\leq b_{k+1}} \frac{dt_1}{t_1-b_1}\cdots\frac{dt_k}{t_k-b_k}$$

The value of the integral above only depends on the homotopy class of the path connecting 0 and a_{m+1} on $\mathbb{C}\setminus\{a_1,\ldots,a_m\}$. To be concrete, when possible, we will integrate over the real line.

It is easy to see that,

$$\begin{aligned}
\mathrm{I}_{n_1,\ldots,n_m}(a_1 : \cdots : a_m : a_{m+1}) &= (-1)^m \mathrm{Li}_{n_1,\ldots,n_m}\left(\frac{a_2}{a_1}, \frac{a_3}{a_2}, \ldots, \frac{a_m}{a_{m-1}}, \frac{a_{m+1}}{a_m}\right) \\
\mathrm{Li}_{n_1,\ldots,n_m}(x_1,\ldots,x_m) &= (-1)^m \mathrm{I}_{n_1,\ldots,n_m}((x_1\ldots x_m)^{-1} : \cdots : x_m^{-1} : 1)
\end{aligned}$$

which gives an analytic continuation to multiple polylogarithms. For instance, with the above convention about integrating over a real segment, simple polylogarithms have an analytic continuation to $\mathbb{C}\setminus[1,\infty)$.

***Exercise* 19.** Prove the previous equations.

***Exercise* 20.** Prove that

$$\zeta(n_1,\ldots,n_m) = \zeta(\underbrace{1,\ldots,1}_{n_m-2}, 2, \ldots, \underbrace{1,\ldots,1}_{n_1-2}, 2).$$

4.1. **Applications of polylogarithms to Mahler measure.** Many of the Mahler measure formulas can be proved via polylogarithms. For instance, formula (3.3) may be rewritten as

$$\mathrm{m}(1+x+y) = \frac{1}{2\pi}(D(e^{2\pi i/3}) - D(e^{-2\pi i/3})) = \frac{1}{2\pi i}(\mathrm{Li}_2(e^{2\pi i/3}) - \mathrm{Li}_2(e^{-2\pi i/3})).$$

Similarly, the Riemann zeta funcion arises as a special value of the polylogarithm:

$$\zeta(n) = \mathrm{Li}_n(1)$$

and this identity appears also in Mahler measure formulas.

Here is a more detailed example.

Theorem 4.4. *For $a \in \mathbb{R}_{>0}$,*

$$\mathrm{m}((1+x)+a(1-x)y) = -\frac{i}{\pi}\left(\mathrm{Li}_2\left(ia\right) - \mathrm{Li}_2\left(-ia\right)\right).$$

Proof. We apply multiplicativity of the Mahler measure and then make the change of variable $x = e^{i\theta}$ and notice that $\frac{1-x}{1+x} = -i\tan\left(\frac{\theta}{2}\right)$.

$$\begin{aligned}
\pi\mathrm{m}((1+x)+a(1-x)y) &= \pi\mathrm{m}(1+x) + \pi\mathrm{m}\left(1+a\frac{1-x}{1+x}y\right) \\
&= \frac{1}{2i}\int_{\mathbb{T}^1}\log^+\left|a\frac{1-x}{1+x}\right|\frac{dx}{x} \\
&= \frac{1}{2}\int_0^{2\pi}\log^+\left|a\tan\left(\frac{\theta}{2}\right)\right|d\theta.
\end{aligned}$$

Now make $w = \frac{1}{a\tan\left(\frac{\theta}{2}\right)}$, then $d\theta = \frac{2\,a\,dw}{1+a^2w^2}$. The previous equation equals

$$\begin{aligned}
& -\int_0^1 \log w\,\frac{2\,a\,dw}{w^2a^2+1} \\
=\; & i\int_0^1\int_w^1 \frac{ds}{s}\left(\frac{1}{w+\frac{i}{a}} - \frac{1}{w-\frac{i}{a}}\right)dw \\
=\; & i\left(\mathrm{I}_2\left(-\frac{i}{a}:1\right) - \mathrm{I}_2\left(\frac{i}{a}:1\right)\right) \\
=\; & -i\left(\mathrm{Li}_2\left(ia\right) - \mathrm{Li}_2\left(-ia\right)\right).
\end{aligned}$$

☺

***Exercise* 21.** Prove that for $a \in \mathbb{C}^\times$,

$$\mathrm{m}(1+x+ay+az) = \begin{cases} \frac{2}{\pi^2}(\mathrm{Li}_3(|a|) - \mathrm{Li}_3(-|a|) & |a| \leq 1, \\ \\ \log|a| + \frac{2}{\pi^2}(\mathrm{Li}_3(|a|^{-1}) - \mathrm{Li}_3(-|a|^{-1}) & |a| \geq 1. \end{cases}$$

***Exercise* 22.** Prove that

$$\mathrm{m}((1+x_1)(1+x_2)+(1-x_1)(1-x_2)y) = \frac{7}{\pi^2}\zeta(3).$$

Polylogarithms play a crutial role in these type of formulas, relating Mahler measure to special values of zeta functions and L-functions. These relationships provide examples of very general conjectures (Beilinson's conjectures) which play a central role in number theory. These statements include, for example, the Birch–Swinnerton-Dyer conjecture, one of the seven Millenium Prize Problems posed by the Clay Institute with a prize of one million dollars for the solution of each.

It is natural to wonder if these applications of polylogarithms to Mahler measure can be generalized. The key ingredient here is the relationship that can be stablished from the integral to hyperlogarithms. Inspired by this, we will explore other types of integrals which should also fit into the statement of Beilinson's conjectures.[‡]

5. Higher Mahler measure

Definition 5.1. The k-higher Mahler measure of P is defined by

$$\mathrm{m}_k(P) := \int_0^1 \cdots \int_0^1 \log^k \left| P\left(e^{2\pi i\theta_1}, \dots, e^{2\pi i\theta_n}\right)\right| d\theta_1 \cdots d\theta_n.$$

In particular, notice that for $k = 1$ we obtain the classical Mahler measure

$$\mathrm{m}_1(P) = \mathrm{m}(P),$$

and

$$\mathrm{m}_0(P) = 1.$$

The simplest example of higher Mahler measure is with the polynomial $P = 1 - x$ [KLO08]:

Theorem 5.2. *([KLO08]) We have,*

$$\begin{aligned}
\mathrm{m}_2(1-x) &= \frac{\zeta(2)}{2} = \frac{\pi^2}{12},\\
\mathrm{m}_3(1-x) &= -\frac{3\zeta(3)}{2},\\
\mathrm{m}_4(1-x) &= \frac{3\zeta(2)^2 + 21\zeta(4)}{4} = \frac{19\pi^4}{240},\\
\mathrm{m}_5(1-x) &= -\frac{15\zeta(2)\zeta(3) + 45\zeta(5)}{2},\\
\mathrm{m}_6(1-x) &= \frac{930\zeta(6) + 180\zeta(3)^2 + 315\zeta(2)\zeta(4) + 15\zeta(2)^3}{8}\\
&= \frac{45}{2}\zeta(3)^2 + \frac{275}{1344}\pi^6,
\end{aligned}$$

and similar equations for higher indexes.

Before proceeding into the proof of Theorem 5.2 we recall some properties of the Gamma function

$$\Gamma(s) := \int_0^\infty t^{s-1}e^{-t}dt.$$

***Exercise* 23.** Prove $\Gamma(1) = 1$ and that $\Gamma(s+1) = s\Gamma(s)$. Deduce that $\Gamma(n+1) = n!$ for n positive integer.

We notice that

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{-1/2}e^{-t}dt = 2\int_0^\infty e^{-s^2}ds = \sqrt{\pi},$$

where we have made the change $t = s^2$ and used the Gaussian integral.

Let

$$B(r,s) = \int_0^1 t^{r-1}(1-t)^{s-1}dt,$$

the Beta function.

***Exercise* 24.** Prove that, whenever the integral for the Beta function converges,

$$B(r,s) = \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)}.$$

Now,

$$B(s,s) = \int_0^1 t^{s-1}(1-t)^{s-1}dt = \frac{1}{2}\int_{-1}^1 \left(\frac{1+u}{2}\right)^{s-1}\left(\frac{1-u}{2}\right)^{s-1}du,$$

[‡]The relationship between higher Mahler measure and Beilinson's conjectures is yet to be stablished. One of the motivations to find examples of higher Mahler measure formulas is precisely the search of a precise formulation of this relationship.

where we have set $t = \frac{1+u}{2}$. This yields,

$$\frac{1}{2^{2s-1}} \int_{-1}^{1} (1-u^2)^{s-1} du = \frac{1}{2^{2s-2}} \int_{0}^{1} (1-u^2)^{s-1} du.$$

By setting $v = u^2$, this equals

$$\frac{1}{2^{2s-1}} \int_{0}^{1} (1-v)^{s-1} v^{-1/2} dv = \frac{1}{2^{2s-1}} B\left(\frac{1}{2}, s\right).$$

Thus,

$$\Gamma(2s) = \frac{2^{2s-1}}{\Gamma\left(\frac{1}{2}\right)} \Gamma(s)\Gamma\left(s+\frac{1}{2}\right) = \frac{2^{2s-1}}{\sqrt{\pi}} \Gamma(s)\Gamma\left(s+\frac{1}{2}\right).$$

The above is called the duplication formula for Γ.

Finally, we state the Weierstrass product without proof.

$$\Gamma(s) = \frac{e^{-\gamma s}}{s} \prod_{n=1}^{\infty} \left(1+\frac{s}{n}\right)^{-1} e^{\frac{s}{n}},$$

where $\gamma = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log n\right) = 0.577216...$ is the Euler–Mascheroni constant.

Proof. [Theorem 5.2] To prove the above equalities one uses a construction by Akatsuka [A09]. We consider the integral

$$Z(s; x-1) = \int_{0}^{1} \left|e^{2\pi i\theta} - 1\right|^{s} d\theta.$$

(This is called the Zeta Mahler measure.) We first make the change $t = \sin^2 \pi\theta$:

$$\begin{aligned} Z(s, x-1) &= 2^{s+1} \int_{0}^{1/2} (\sin \pi\theta)^s d\theta \\ &= \frac{2^s}{\pi} \int_{0}^{1} t^{\frac{s-1}{2}} (1-t)^{-1/2} dt. \end{aligned}$$

Thus, we obtain the Beta function

$$\begin{aligned} &= \frac{2^s}{\pi} B\left(\frac{s+1}{2}, \frac{1}{2}\right) \\ &= \frac{2^s}{\pi} \frac{\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{s}{2}+1\right)} \\ &= \frac{2^s}{\sqrt{\pi}} \frac{\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s}{2}+1\right)}, \end{aligned}$$

where $\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt$ is the Gamma function. Hence, by using

$$\Gamma\left(\frac{s+1}{2}\right) = \frac{\Gamma(s)}{\Gamma\left(\frac{s}{2}\right)} 2^{1-s} \pi^{\frac{1}{2}} = \frac{\Gamma(s+1)}{\Gamma\left(\frac{s}{2}+1\right)} 2^{-s} \pi^{\frac{1}{2}},$$

and

$$\Gamma(s+1)^{-1} = e^{\gamma s} \prod_{n=1}^{\infty} \left(1+\frac{s}{n}\right) e^{-\frac{s}{n}},$$

we obtain

$$\begin{aligned}
Z(s, x-1) &= \prod_{n=1}^{\infty} \frac{\left(1+\frac{s}{2n}\right)^2}{1+\frac{s}{n}} \\
&= \exp\left(\sum_{n=1}^{\infty}\left\{2\log\left(1+\frac{s}{2n}\right)-\log\left(1+\frac{s}{n}\right)\right\}\right) \\
&= \exp\left(\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\sum_{n=1}^{\infty}\left\{2\left(\frac{1}{2n}\right)^k-\frac{1}{n^k}\right\}s^k\right) \\
&= \exp\left(\sum_{k=2}^{\infty}\frac{(-1)^{k-1}}{k}\zeta(k)(2^{1-k}-1)s^k\right) \\
&= \exp\left(\sum_{k=2}^{\infty}\frac{(-1)^{k}(1-2^{1-k})\zeta(k)}{k}s^k\right).
\end{aligned}$$

The Zeta Mahler measure yields a Taylor series whose coefficients are given by $\mathrm{m}_k(P)$.

$$\begin{aligned}
\sum_{k=0}^{\infty}\frac{\mathrm{m}_k(x-1)}{k!}s^k &= Z(s,x-1) \\
&= \exp\left(\sum_{k=2}^{\infty}\frac{(-1)^{k}(1-2^{1-k})\zeta(k)}{k}s^k\right).
\end{aligned}$$

☺

***Exercise* 25.** Prove that

$$\mathrm{m}_k(1-x) = \sum_{b_1+\cdots+b_h=k,\, b_i\geq 2}\frac{(-1)^k k!}{2^{2h}}\zeta(b_1,\ldots,b_h),$$

where $\zeta(b_1,\ldots,b_h)$ denotes

$$\zeta(b_1,\ldots,b_h) = \sum_{l_1<\cdots<l_h}\frac{1}{l_1^{b_1}\ldots l_h^{b_h}}.$$

Hint:

$$\log^k|1-x| = (\mathrm{Re}\log(1-x))^k = \left(\frac{1}{2}(\log(1-x)+\log(1-x^{-1}))\right)^k$$

$$= \frac{1}{2^k}\left(\int_0^1\frac{dt}{t-x^{-1}}+\int_0^1\frac{dt}{t-x}\right)^k = \frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}\left(\int_0^1\frac{dt}{t-x^{-1}}\right)^j\left(\int_0^1\frac{dt}{t-x}\right)^{k-j}.$$

We know how to answer Lehmer's question in the case of the higher Mahler measure.

Theorem 5.3. *([LS11]) If $P(x)\in\mathbb{Z}[x]$ is not a monomial, then for any $h\geq 1$,*

$$\mathrm{m}_{2h}(P) \geq \begin{cases}\left(\frac{\pi^2}{12}\right)^h, & \text{if } P(x) \text{ is reciprocal,}\\ \left(\frac{\pi^2}{48}\right)^h, & \text{if } P(x) \text{ is non-reciprocal.}\end{cases}$$

Let $P_n(x)=\frac{x^n-1}{x-1}$. For $h\geq 1$ fixed,

$$\lim_{n\to\infty}\mathrm{m}_{2h+1}(P_n)=0.$$

Moreover, this sequence is nonconstant.

A natural generalization for the k-higher Mahler measure is the multiple higher Mahler measure for more than one polynomial.

Definition 5.4. Let $P_1, \dots, P_k \in \mathbb{C}[x_1^{\pm}, \dots, x_n^{\pm}]$ be non-zero Laurent polynomials. Their multiple higher Mahler measure is defined by

$$\begin{aligned} & \mathrm{m}(P_1, \dots, P_k) \\ := & \int_0^1 \cdots \int_0^1 \log\left|P_1\left(e^{2\pi i\theta_1}, \cdots, e^{2\pi i\theta_n}\right)\right| \cdots \log\left|P_k\left(e^{2\pi i\theta_1}, \cdots, e^{2\pi i\theta_n}\right)\right| d\theta_1 \cdots d\theta_n. \end{aligned}$$

This construction yields the higher Mahler measure of one polynomial as a special case:

$$\mathrm{m}_k(P) = \mathrm{m}(\underbrace{P, \dots, P}_{k}).$$

Moreover, the above definition implies that

$$\mathrm{m}(P_1) \cdots \mathrm{m}(P_k) = \mathrm{m}(P_1, \dots, P_k)$$

when the variables of P_j's are algebraically independent.

The simplest example is this regard is, again, given by linear polynomials.

Theorem 5.5. *([KLO08]) For* $0 \le \alpha \le 1$

$$\mathrm{m}(1-x, 1-e^{2\pi i\alpha}x) = \frac{\pi^2}{2}\left(\alpha^2 - \alpha + \frac{1}{6}\right).$$

Proof. Let $A_\varepsilon = [\varepsilon, \alpha - \varepsilon] \cup [\alpha + \varepsilon, 1 - \varepsilon]$. By definition,

$$\begin{aligned} \mathrm{m}(1-x, 1-e^{2\pi i\alpha}x) &= \int_0^1 \operatorname{Re}\log(1-e^{2\pi i\theta}) \cdot \operatorname{Re}\log(1-e^{2\pi i(\theta+\alpha)}) d\theta \\ &= \int_0^1 \left(-\sum_{k=1}^{\infty} \frac{1}{k}\cos 2\pi k\theta\right)\left(-\sum_{l=1}^{\infty}\frac{1}{l}\cos 2\pi l(\theta+\alpha)\right) d\theta \\ &= \sum_{k,l\ge 1} \frac{1}{kl}\int_{A_\varepsilon} \cos(2\pi k\theta)\cos(2\pi l(\theta+\alpha)) d\theta \\ &\quad + \int_{[0,1]\setminus A_\varepsilon}\left(-\sum_{k=1}^{\infty} \frac{1}{k}\cos 2\pi k\theta\right)\left(-\sum_{l=1}^{\infty}\frac{1}{l}\cos 2\pi l(\theta+\alpha)\right) d\theta. \end{aligned}$$

Because $\log|1-e^{2\pi i\theta}| = \log|2\sin\pi\theta| \sim \log|2\pi\theta|$ for θ near zero, and $\int_0^\varepsilon \log(Kx)dx \to 0$ as $\varepsilon \to 0$, the last term approaches zero.

Notice that

$$\int_{[0,1]\setminus A_\varepsilon} \cos(2\pi k\theta)\cos(2\pi l(\theta+\alpha))d\theta = \begin{cases} O(\varepsilon) & \text{if } l = k, \\ \frac{O(\varepsilon)}{|k-l|} & \text{otherwise,} \end{cases}$$

and

$$\int_0^1 \cos(2\pi k\theta)\cos(2\pi l(\theta+\alpha))d\theta = \begin{cases} \frac{1}{2}\cos(2\pi k\alpha) & \text{if } l = k, \\ 0 & \text{otherwise.} \end{cases}$$

By putting everything together we conclude that

$$\mathrm{m}(1-x, 1-e^{2\pi i\alpha}x) = \frac{1}{2}\sum_{k=1}^{\infty}\frac{\cos(2\pi k\alpha)}{k^2} = \frac{\pi^2}{2}\left(\alpha^2 - \alpha + \frac{1}{6}\right).$$

☺

More generally,

$$
\mathrm{m}(1-\alpha x, 1-\beta x) = \begin{cases} \frac{1}{2}\operatorname{Re}\operatorname{Li}_2\left(\alpha\bar{\beta}\right) & \text{if } |\alpha|, |\beta| \leq 1, \\ \\ \frac{1}{2}\operatorname{Re}\operatorname{Li}_2\left(\frac{\alpha\beta}{|\alpha|^2}\right) & \text{if } |\alpha| \geq 1, |\beta| \leq 1, \\ \\ \frac{1}{2}\operatorname{Re}\operatorname{Li}_2\left(\frac{\alpha\bar{\beta}}{|\alpha\beta|^2}\right) + \log|\alpha|\log|\beta| & \text{if } |\alpha|, |\beta| \geq 1. \end{cases} \tag{5.1}
$$

yields the equivalent of a Jensen's formula for multiple Mahler measure:

$$
\mathrm{m}(1-\alpha x) = \begin{cases} 0 & \text{if } |\alpha| \leq 1, \\ \\ \log|\alpha| & \text{if } |\alpha| \geq 1. \end{cases}
$$

***Exercise* 26.** Prove Equation (5.1).

***Exercise* 27.** Prove that

$$
\begin{aligned}
\mathrm{m}(1-x, 1-e^{2\pi i\alpha}x, 1-e^{2\pi i\beta}x) \quad = \quad & -\frac{1}{4}\sum_{k,l\geq 1}\frac{\cos 2\pi((k+l)\beta - l\alpha)}{kl(k+l)} \\
& -\frac{1}{4}\sum_{k,m\geq 1}\frac{\cos 2\pi((k+m)\alpha - m\beta)}{km(k+m)} \\
& -\frac{1}{4}\sum_{l,m\geq 1}\frac{\cos 2\pi(l\alpha + m\beta)}{lm(l+m)}.
\end{aligned}
$$

Multiple higher Mahler measure has applications to the computation of the higher Mahler measure.
For example, Theorem 5.3 is proven as a consequence of the following result.

Theorem 5.6. *([LS11])If $P(x) \in \mathbb{Z}[x]$ is reciprocal, then*

$$
\mathrm{m}_2(P) \geq \frac{\pi^2}{12}.
$$

This inequality is sharp, with equality for $x-1$.

Proof. (Theorem 5.3 from Theorem 5.6) First suppose that $h=2$ and $P(x)$ non-reciprocal. Let $d = \deg P$, and consider $P^*(x) = x^d P(x^{-1})$. Thus $P(x)P^*(x) \in \mathbb{Z}[x]$ is reciprocal. Moreover, $\mathrm{m}_2(P) = \mathrm{m}_2(P^*) = \mathrm{m}(P, P^*)$, thus,

$$
\mathrm{m}_2(PP^*) = \mathrm{m}_2(P) + 2\mathrm{m}(P, P^*) + \mathrm{m}_2(P^*) = 4\mathrm{m}_2(P).
$$

We obtain the desired bound by applying Theorem 5.6 to PP^*.

The rest of the proof (for $h > 2$) is left as an exercise. ☺

***Exercise* 28.** Complete the proof of Theorem 5.3 by proving

(a)

$$
\mathrm{m}_{2h}(P) \geq \mathrm{m}_2(P)^h,
$$

(b)

$$
\mathrm{m}_{2h}(P) \geq \mathrm{m}(P)^{2h}.
$$

***Open Question* 1.** Is it possible to improve the bounds in Theorem 5.3?

***Exercise* 29.** With the help of a computer explore the values of the higher Mahler measure in polynomials. For example, compute m_2 for the polynomials in Exercise 7.

***Open Question* 2.** Can you find $P \in \mathbb{Z}[x]$ non-reciprocal with $\mathrm{m}_2(P) < \mathrm{m}_2(x^3+x+1) = 0.3275495729\ldots$? (Notice that $\frac{\pi^2}{48} = 0.2056167583\ldots$.)

Theorem 5.7. *([KLO08])*

$$\begin{aligned}\mathrm{m}_2(1-x+y(1+x)) \quad = \quad & \frac{4i}{\pi}(\mathrm{Li}_{2,1}(-i,-i) - \mathrm{Li}_{2,1}(i,i)) \\ & +\frac{6i}{\pi}(\mathrm{Li}_{2,1}(i,-i) - \mathrm{Li}_{2,1}(-i,i)) \\ & +\frac{i}{\pi}(\mathrm{Li}_{2,1}(1,-i) - \mathrm{Li}_{2,1}(1,i)) - \frac{7\zeta(2)}{16} + \frac{\log 2}{\pi} L(\chi_{-4},2)\end{aligned}$$

Proof. First notice that

$$\mathrm{m}_2(1-x+y(1+x)) = \mathrm{m}_2\left(\left(\frac{1-x}{1+x}\right)+y\right) + 2\mathrm{m}\left(\left(\frac{1-x}{1+x}\right)+y, 1+x\right) + \mathrm{m}_2(1+x). \tag{5.2}$$

For the first term, we have

$$\mathrm{m}_2\left(\left(\frac{1-x}{1+x}\right)+y\right) = \frac{1}{(2\pi i)^2}\int_{|y|=1}\int_{|x|=1} \log^2\left|\left(\frac{1-x}{1+x}\right)+y\right|\frac{dx}{x}\frac{dy}{y}.$$

By applying (5.1), the above line becomes

$$\begin{aligned} = \quad & \frac{1}{2\pi i}\int_{|x|=1,\,|1-x|\le|1+x|} \frac{1}{2}\mathrm{Li}_2\left(\left|\frac{1-x}{1+x}\right|^2\right)\frac{dx}{x} + \frac{1}{2\pi i}\int_{|x|=1,\,|1-x|\ge|1+x|}\frac{1}{2}\mathrm{Li}_2\left(\left|\frac{1+x}{1-x}\right|^2\right)\frac{dx}{x} \\ & + \frac{1}{2\pi i}\int_{|x|=1,\,|1-x|\ge|1+x|}\log^2\left|\frac{1-x}{1+x}\right|\frac{dx}{x} \\ = \quad & \frac{1}{2\pi i}\int_{|x|=1,\,|1-x|\le|1+x|}\mathrm{Li}_2\left(\left|\frac{1-x}{1+x}\right|^2\right)\frac{dx}{x} + \frac{1}{2\pi i}\int_{|x|=1,\,|1-x|\ge|1+x|}\log^2\left|\frac{1-x}{1+x}\right|\frac{dx}{x}.\end{aligned}$$

For the second term in equation (5.2) we obtain

$$\mathrm{m}\left(\left(\frac{1-x}{1+x}\right)+y, 1+x\right) = \frac{1}{(2\pi i)^2}\int_{|y|=1}\int_{|x|=1}\log\left|\left(\frac{1-x}{1+x}\right)+y\right|\log|1+x|\frac{dx}{x}\frac{dy}{y}.$$

By Jensen's formula respect to the variable y,

$$= \frac{1}{2\pi i}\int_{|x|=1}\log^+\left|\frac{1-x}{1+x}\right|\log|1+x|\frac{dx}{x} = \frac{1}{2\pi i}\int_{|x|=1,\,|1-x|\ge|1+x|}\log\left|\frac{1-x}{1+x}\right|\log|1+x|\frac{dx}{x}.$$

Then equation (5.2) becomes

$$\begin{aligned}\mathrm{m}_2(1-x+y(1+x)) \quad = \quad & \frac{1}{2\pi i}\int_{|x|=1,\,|1-x|\le|1+x|}\mathrm{Li}_2\left(\left|\frac{1-x}{1+x}\right|^2\right)\frac{dx}{x} \\ & +\frac{1}{2\pi i}\int_{|x|=1,\,|1-x|\ge|1+x|}(\log^2|1-x| - \log^2|1+x|)\frac{dx}{x} \\ & +\frac{\zeta(2)}{2}.\end{aligned} \tag{5.3}$$

For the first term in (5.3), set $x = e^{2i\theta}$,

$$\begin{aligned}& \frac{1}{2\pi i}\int_{|x|=1,\,|1-x|\le|1+x|}\mathrm{Li}_2\left(\left|\frac{1-x}{1+x}\right|^2\right)\frac{dx}{x} \\ = \quad & \frac{2}{\pi}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\mathrm{Li}_2\left(\tan^2\theta\right)d\theta = \frac{4}{\pi}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(\mathrm{Li}_2(\tan\theta) + \mathrm{Li}_2(-\tan\theta))d\theta.\end{aligned}$$

Now we make the change of variables $y = \tan\theta$.

$$\begin{aligned}
&= \frac{8}{\pi}\int_0^1 (\mathrm{Li}_2(y) + \mathrm{Li}_2(-y))\frac{dy}{y^2+1}\\
&= \frac{4}{\pi}\int_0^1 (\mathrm{Li}_2(y) + \mathrm{Li}_2(-y))\left(\frac{1}{1+iy} + \frac{1}{1-iy}\right)dy\\
&= \frac{4}{\pi}(i\mathrm{Li}_{2,1}(i,-i) + i\mathrm{Li}_{2,1}(-i,-i) - i\mathrm{Li}_{2,1}(-i,i) - i\mathrm{Li}_{2,1}(i,i)).
\end{aligned}$$

For the second term in (5.3),

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{|x|=1,\,|1-x|\geq|1+x|} (\log^2|1-x| - \log^2|1+x|)\frac{dx}{x}\\
&= \sum_{k,l\geq 1}\frac{1-(-1)^{k+l}}{kl}2\int_{\frac{1}{4}}^{\frac{3}{4}}\cos(2\pi k\theta)\cos(2\pi l\theta)d\theta\\
&= \sum_{k,l\geq 1}\frac{1-(-1)^{k+l}}{2\pi kl}\left(\frac{i^{k+l+1}(1-(-1)^{k+l})}{k+l} + \frac{i^{k-l+1}(1-(-1)^{k-l})}{k-l}\right)\\
&= \frac{i}{\pi}\sum_{k,l\geq 1}\frac{(1-(-1)^{k+l})i^{k+l}}{kl^2} - \frac{i}{\pi}\sum_{k,l\geq 1}\frac{(1-(-1)^{k+l})i^{k+l}}{(k+l)l^2}\\
&\quad + \frac{2i}{\pi}\sum_{k>l\geq 1}\frac{(1-(-1)^{k+l})i^{k-l}}{(k-l)l^2} - \frac{2i}{\pi}\sum_{k>l\geq 1}\frac{(1-(-1)^{k+l})i^{k-l}}{kl^2}\\
&= \frac{i}{\pi}(\mathrm{Li}_1(i)\mathrm{Li}_2(i) - \mathrm{Li}_1(-i)\mathrm{Li}_2(-i) - \mathrm{Li}_{2,1}(1,i) + \mathrm{Li}_{2,1}(1,-i))\\
&\quad + \frac{2i}{\pi}(\zeta(2)(\mathrm{Li}_1(i) - \mathrm{Li}_1(-i)) - \mathrm{Li}_{2,1}(-i,i) + \mathrm{Li}_{2,1}(i,-i))\\
&= \frac{i}{\pi}(-i\log 2L(\chi_{-4},2) - \frac{\pi i}{16}\zeta(2) - \mathrm{Li}_{2,1}(1,i) + \mathrm{Li}_{2,1}(1,-i))\\
&\quad + \frac{2i}{\pi}(\zeta(2)\frac{\pi i}{2} - \mathrm{Li}_{2,1}(-i,i) + \mathrm{Li}_{2,1}(i,-i)).
\end{aligned}$$

Putting everything together in (5.3), we obtain the final result

$$\begin{aligned}
&\mathrm{m}_2(1-x+y(1+x))\\
&= \frac{4i}{\pi}(\mathrm{Li}_{2,1}(-i,-i) - \mathrm{Li}_{2,1}(i,i)) + \frac{6i}{\pi}(-\mathrm{Li}_{2,1}(-i,i) + \mathrm{Li}_{2,1}(i,-i))\\
&\quad + \frac{i}{\pi}(-\mathrm{Li}_{2,1}(1,i) + \mathrm{Li}_{2,1}(1,-i)) - \frac{7\zeta(2)}{16} + \frac{\log 2}{\pi}L(\chi_{-4},2).
\end{aligned}$$

☺

This result may be contrasted to Smyth's

$$\mathrm{m}(1-x+y(1+x)) = \frac{2}{\pi}L(\chi_{-4},2).$$

***Exercise* 30.** Prove the above formula from Theorem 4.4.

Naturally, we have the following generalization of the Zeta Mahler measure.

Definition 5.8. Let $P_1,\dots,P_k \in \mathbb{C}[x_1^{\pm},\dots,x_n^{\pm}]$ be non-zero Laurent polynomials. Their higher zeta Mahler measure is defined by

$$\begin{aligned}
&Z(s_1,\dots,s_l;P_1,\dots,P_l)\\
&= \int_0^1\cdots\int_0^1 \left|P_1\left(e^{2\pi i\theta_1},\cdots,e^{2\pi i\theta_r}\right)\right|^{s_1}\cdots\left|P_l\left(e^{2\pi i\theta_1},\cdots,e^{2\pi i\theta_r}\right)\right|^{s_l}d\theta_1\cdots d\theta_r,
\end{aligned}$$

Its Taylor coefficients are related to the multiple higher Mahler measure:

$$\frac{\partial^l}{\partial s_1 \cdots \partial s_l} Z(0,\dots,0;P_1,\dots,P_l) = \mathrm{m}(P_1,\dots,P_l).$$

Similarly to the case of $x-1$, one gets

$$\begin{aligned} Z(s,t;x-1,x+1) &= \int_0^1 |2\sin \pi\theta|^s \, |2\cos \pi\theta|^t \, d\theta \\ &= 2^{s+t+1}\int_0^{1/2} (\sin \pi\theta)^s (\cos \pi\theta)^t d\theta. \end{aligned}$$

Setting $u = \sin^2(\pi\theta)$,

$$\begin{aligned} Z(s,t;x-1,x+1) &= \frac{2^{s+t}}{\pi}\int_0^1 u^{\frac{s-1}{2}}(1-u)^{\frac{t-1}{2}} du \\ &= \frac{2^{s+t}}{\pi}\frac{\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{t+1}{2}\right)}{\Gamma\left(\frac{s+t}{2}+1\right)} \\ &= \frac{\Gamma(s+1)\Gamma(t+1)}{\Gamma\left(\frac{s}{2}+1\right)\Gamma\left(\frac{t}{2}+1\right)\Gamma\left(\frac{s+t}{2}+1\right)} \\ &= \prod_{n=1}^{\infty} \frac{\left(1+\frac{s}{2n}\right)\left(1+\frac{t}{2n}\right)\left(1+\frac{s+t}{2n}\right)}{\left(1+\frac{s}{n}\right)\left(1+\frac{t}{n}\right)} \\ &= \exp\left(\sum_{k=2}^{\infty}\frac{(-1)^k}{k}\zeta(k)\left\{(1-2^{-k})(s^k+t^k) - 2^{-k}(s+t)^k\right\}\right) \end{aligned}$$

around $s=t=0$.

Thus,

$$\mathrm{m}(\underbrace{x-1,\dots,x-1}_{k},\underbrace{x+1,\dots,x+1}_{l}) = \int_0^1 (\log|2\sin\pi\theta|)^k (\log|2\cos\pi\theta|)^l d\theta$$

belongs to $\mathbb{Q}[\pi^2, \zeta(3), \zeta(5), \zeta(7), \dots]$ for integers $k, l \geq 0$.

We obtain, for instance, the following equalities.

$$\begin{aligned} \mathrm{m}(x-1,x+1) &= \int_0^1 \log|2\sin\pi\theta| \log|2\cos\pi\theta| \, d\theta = -\frac{\zeta(2)}{4} = -\frac{\pi^2}{24}, \\ \mathrm{m}(x-1,x-1,x+1) &= \int_0^1 (\log|2\sin\pi\theta|)^2 \log|2\cos\pi\theta| \, d\theta = 2\frac{\zeta(3)}{8} = \frac{\zeta(3)}{4}, \\ \mathrm{m}(x-1,x+1,x+1) &= \int_0^1 \log|2\sin\pi\theta| \, (\log|2\cos\pi\theta|)^2 d\theta = 2\frac{\zeta(3)}{8} = \frac{\zeta(3)}{4}. \end{aligned}$$

***Exercise* 31.** Prove the following.

(a) For a positive constant λ, we have $Z(s, \lambda P) = \lambda^s Z(s,P)$.

(b) Let $P \in \mathbb{C}[x_1^{\pm 1},\dots,x_n^{\pm 1}]$ be a Laurent polynomial such that it takes non-negative real values in the unit torus. Then we have the following series expansion on $|\lambda| \leq 1/\max(P)$, where $\max(P)$ is the maximum of P on the unit torus:

$$\begin{aligned} Z(s,1+\lambda P) &= \sum_{k=0}^{\infty}\binom{s}{k} Z(k,P)\lambda^k, \\ \mathrm{m}(1+\lambda P) &= \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k} Z(k,P)\lambda^k. \end{aligned}$$

More generally,

$$\mathrm{m}_j(1+\lambda P) = j!\sum_{0<k_1<\cdots<k_j}\frac{(-1)^{k_j-j}}{k_1\dots k_j}Z(k_j,P)\lambda^{k_j}.$$

(c) $Z(s,P)=Z(\frac{s}{2},P\bar{P})$, where we put $\bar{P}=\sum_\alpha \bar{a}_\alpha x^{-\alpha}$ for $P=\sum_\alpha a_\alpha x^\alpha$. Note that $P\bar{P}$ is real-valued on the torus.

Theorem 5.9. *([KLO08]) Let $c\geq 2$.*

- $$\begin{aligned} Z(s,x+y+c) &= c^s\sum_{j=0}^\infty \binom{s/2}{j}^2\frac{1}{c^{2j}}\binom{2j}{j} \\ &= c^s\,{}_3F_2\left(\begin{matrix}-\frac{s}{2},-\frac{s}{2},\frac12\\ 1,1\end{matrix}\,\middle|\,\frac{4}{c^2}\right), \end{aligned}$$

  where the generalized hypergeometric series ${}_3F_2$ is defined by

  $${}_3F_2\left(\begin{matrix}a_1,a_2,a_3\\ b_1,b_2\end{matrix}\,\middle|\,z\right)=\sum_{j=0}^\infty\frac{(a_1)_j(a_2)_j(a_3)_j}{(b_1)_j(b_2)_j j!}z^j,$$

  with the Pochhammer symbol defined by $(a)_j=a(a+1)\cdots(a+j-1)$.
- $$\mathrm{m}_2(x+y+c)=\log^2 c+\frac12\sum_{k=1}^\infty\binom{2k}{k}\frac{1}{k^2c^{2k}}.$$
- $$\mathrm{m}_3(x+y+c)=\log^3 c+\frac32\log c\sum_{k=1}^\infty\binom{2k}{k}\frac{1}{k^2c^{2k}}-\frac32\sum_{k=2}^\infty\binom{2k}{k}\frac{1}{k^2c^{2k}}\sum_{j=1}^{k-1}\frac1j.$$

In particular, we obtain the special values

- $$\mathrm{m}_2(x+y+2)=\frac{\zeta(2)}{2},$$
- $$\mathrm{m}_3(x+y+2)=\frac92\log 2\zeta(2)-\frac{15}{4}\zeta(3).$$

The previous Theorem may be completed with the trivial statement

$$\mathrm{m}(x+y+2)=\log 2.$$

In fact, the motivation for setting $c=2$ is that this is the precise point where the family of polynomials $x+y+c$ reaches the unit torus singularly. In classical Mahler measure, those polynomials are among the simplest to compute the Mahler measure, and the same is true in higher Mahler measures.

6. Log-sine integrals and Mahler measure

We consider the works of Borwein, Straub and collaborators [BS12, BBSW12, BS11].

Definition 6.1. For n a positive integer and k a non-negative integer, the generalized log-sine integral is defined by

$$\mathrm{Ls}_n^{(k)}(\sigma):=-\int_0^\sigma\theta^k\log^{n-1-k}\left|2\sin\frac{\theta}{2}\right|d\theta.$$

This integral was studied by Lewin who gave several evaluations [Le81]. We use the notation $\mathrm{Ls}_n(\sigma)$ for the case $k=0$.

Some special values are expressed in terms of polylogarithms, such as

$$-\frac1\pi\sum_{m=0}^\infty\mathrm{Ls}_{m+1}(\pi)\frac{\lambda^m}{m!}=\frac{\Gamma(1+\lambda)}{\Gamma(1+\lambda/2)^2},$$

which implies the recurrence

$$\frac{(-1)^n}{n!}\mathrm{Ls}_{n+2}(\pi) = \pi(1-2^{-n})\zeta(n+1) + \sum_{k=1}^{n-2}\frac{(-1)^k}{(k+1)!}(1-2^{1+k-n})\zeta(n-k)\mathrm{Ls}_{k+2}(\pi),$$

and yields, starting from $\mathrm{Ls}_2(\pi)=0$,

$$\begin{aligned}
\mathrm{Ls}_3(\pi) &= -\frac{\pi^3}{12},\\
\mathrm{Ls}_4(\pi) &= \frac{3\pi}{2}\zeta(3),\\
\mathrm{Ls}_5(\pi) &= -\frac{19}{240}\pi^5,\\
\mathrm{Ls}_4(\pi) &= \frac{45\pi}{2}\zeta(5)+\frac{5}{4}\pi^3\zeta(3).
\end{aligned}$$

More generally, we have the following result from [BS11].

Theorem 6.2. *([BS11]) For $0\le\tau\le 2\pi$, and nonnegative integers n,k such that $n-k\ge 2$,*

$$\begin{aligned}
&\zeta(\{1\}^k, n-k) - \sum_{j=0}^{k}\frac{(-i\tau)^j}{j!}\mathrm{Li}_{\{1\}^{n-k-2},2+k-j}(\{1\}^{n-k-2},e^{i\tau})\\
&= \frac{i^{k+1}(-1)^{n-1}}{(n-1)!}\sum_{r=0}^{n-k-1}\sum_{m=0}^{r}\binom{n-1}{k,m,r-m}\left(\frac{i}{2}\right)^r(-\pi)^{r-m}\mathrm{Ls}_{n-(r-m)}^{(k+m)}(\tau).
\end{aligned}$$

Here are some results that were proved using properties of log-sine integrals.

- Let
$$\mathrm{m}_k(1+x+y_*) := \mathrm{m}(1+x+y_1, 1+x+y_2,\dots,1+x+y_k).$$
Then, Sasaki [Sa10] proved
$$\mathrm{m}_k(1+x+y_*) = \frac{1}{\pi}\mathrm{Ls}_{k+1}\left(\frac{\pi}{3}\right) - \frac{1}{\pi}\mathrm{Ls}_{k+1}(\pi).$$
In particular, [BS12] proved
$$\begin{aligned}
\mathrm{m}_2(1+x+y_*) &= \frac{\pi^2}{54},\\
\mathrm{m}_3(1+x+y_*) &= \frac{9}{2\pi}\mathrm{Im}\,(\mathrm{Li}_4(e^{\pi i/3})),\\
\mathrm{m}_4(1+x+y_*) &= \frac{6}{\pi}\mathrm{Im}\,(\mathrm{Li}_{1,4}(1,e^{\pi i/3})) - \frac{\pi^4}{4860},
\end{aligned}$$
and similar formulas por $k=5,6$.
- Let
$$\mathrm{m}_k(1+x+y_*+z_*) := \mathrm{m}(1+x+y_1+z_1, 1+x+y_2+z_2,\dots,1+x+y_k+z_k).$$
Then, in [BS12]:
$$\mathrm{m}_k(1+x+y_*+z_*) = \frac{1}{\pi^{k+1}}\int_0^\pi\left(\theta\log\left(2\sin\frac{\theta}{2}\right)+D(e^{i\theta})\right)^k d\theta.$$
In particular,
$$\begin{aligned}
\mathrm{m}_1(1+x+y_*+z_*) &= \frac{7}{2\pi^2}\zeta(3),\\
\mathrm{m}_2(1+x+y_*+z_*) &= \frac{4}{\pi^2}\mathrm{Li}_{1,3}(1,-1)+\frac{7\pi^2}{360},
\end{aligned}$$

- Other results from [BS12] include

$$\mathrm{m}(1+x, 1+x+y+z) = \frac{2}{\pi^2}\lambda_4\left(\frac{1}{2}\right) - \frac{19}{720}\pi^2,$$
$$\mathrm{m}(1+x, 1+x, 1+x+y+z) = \frac{4}{3\pi^2}\lambda_5\left(\frac{1}{2}\right) - \frac{3}{4}\zeta(3) + \frac{31}{16\pi^2}\zeta(5),$$

  where

$$\lambda_n(x) := (n-2)!\sum_{k=0}^{n-2}\frac{(-1)^k}{k!}\mathrm{Li}_{n-k}(x)\log^k|x| + \frac{(-1)^n}{n}\log^n|x|.$$

- Examples involving higher Mahler measure were computed in [BBSW12]

$$\mathrm{m}_2(1+x+y) = \frac{3}{\pi}\mathrm{Ls}_3\left(\frac{2\pi}{3}\right) + \frac{\pi^2}{4},$$
$$\mathrm{m}_3(1+x+y) \stackrel{?}{=} \frac{6}{\pi}\mathrm{Ls}_4\left(\frac{2\pi}{3}\right) - \frac{9}{\pi}\mathrm{Im}\left(\mathrm{Li}_4(e^{\pi i/3})\right) - \frac{\pi}{4}D(e^{\pi i/3}) - \frac{13}{2}\zeta(3).$$

***Exercise* 32.** Find
$$\mathrm{m}_2\left(\frac{1+x+y}{1+x+z}\right).$$

***Open Question* 3.** Is it possible to express $\mathrm{Ls}_n\left(\frac{2\pi}{3}\right)$ in terms of polylogarithms so that the above formulas only contain polylogarithms? (Perhaps using Theorem 6.2?)

***Open Question* 4.** Investigate identities between log-sine integrals and polylogarithms with the help of the program developed by Borwein and Straub in [BS11].

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7. Some simple SAGE routines

Specify that the polynomials are over $\mathbb{C}$:

```
sage: complexpoly.<x>=PolynomialRing(CC)
sage: load ("mahler.sage")
```

7.1. Mahler measure for one-variable polynomials.

```
def mahler(p):
    m=0
    for rm in p.roots():
        rt=rm[0]
        mul=rm[1]
        if abs(rt)>1:
            m=m+log(abs(rt))*mul
    return(m)
```

7.2. Higher Mahler measure of order 2 for one-variable polynomials.

```
 def mahler2(p):
    m2=0
    for rm1 in p.roots():
        rt1=rm1[0]
        mul1=rm1[1]
        for rm2 in p.roots():
            rt2=rm2[0]
            mul2=rm2[1]
            if abs(rt1)<=1 and abs(rt2)<=1:
                   m2=m2+real(dilog(rt1*conjugate(rt2)))/2*mul1*mul2
            if abs(rt1)<=1 and abs(rt2)>1:
                   m2=m2+real(dilog(rt1/conjugate(rt2)))/2*mul1*mul2
            if abs(rt1)>1 and abs(rt2)<=1:
                   m2=m2+real(dilog(rt2/conjugate(rt1)))/2*mul1*mul2
            if abs(rt1)>1 and abs(rt2)>1:
                   m2=m2+(real(dilog(1/rt2/conjugate(rt1)))/2+log(abs(rt1))*log(abs(rt2)))*mul1*mul2
    return(m2)
```

7.3. General higher Mahler measure for one-variable polynomials.

This sometimes works.

```
sage: x=var('x')
sage: (log(abs(1-exp(I*x)))/2/pi).nintegral(x,0,2*pi)
```

7.4. Δ_n.

```
 def delta(n,p):
     pr=1
     for r in p.roots():
         rt=r[0]
         mul=r[1]
         pr=pr*((rt^n-1)^mul)
     return int(round(real_part(pr)))
```

7.5. Log-sine integrals.

The package `logsine.sage` is available at `http://arminstraub.com/software/lstoli-sage`.

8. Suggestions for the exercises

***Suggestion* 1**. By

$$(a^{rs}-1)=(a^r-1)(a^{(s-1)r}+a^{(s-2)r}+\cdots+a^r+1),$$

we have that $a-1$ is a nontrivial divisor of a^p-1 unless $a-1=1$, in which case $a=2$, or $a^p-1=a-1$, in which case $p=1$.

***Suggestion* 2**. By Fermat's little theorem, $2^{q-1}\equiv 1 \bmod q$. We have, by hypothesis, $2^p\equiv 1 \bmod q$. Since p is prime, we must have that p is the order of 2 modulo q. Therefore, $p \mid q-1$. Since q is also odd, we can write $q=2pk+1$.

***Suggestion* 3**. If n divides m, then $m=kn$ for some $k\geq 1$ and

$$\frac{\Delta_{kn}(P)}{\Delta_n(P)}=\prod_{i=1}^{d}\left(1+\alpha_i^n+\cdots+\alpha_i^{(k-1)n}\right)$$

is a symmetric function of the roots of P, and so is an integer.

***Suggestion* 4**. Fix ξ_n an nth root of unity. If the roots of $P(x)$ are α_i, then the roots of $P(x^n)$ are $\alpha_i^{1/n}\xi_n^k$ for $k=0,\dots,n-1$ and $\alpha_i^{1/n}$ an nth root of α_i. In addition, $|\alpha_i^{1/n}\xi_n^k|=|\alpha_i|^{1/n}$, so $|\alpha_i^{1/n}\xi_n^k|$ and $|\alpha_i|$ are both either >1 or ≤ 1 at the same time. We have

$$M(P(x))=|a|\prod_i\max\{1,|\alpha_i|\}=|a|\prod_i\max\{1,|\alpha_i|^{1/n}\}^n=|a|\prod_i\prod_k\max\{1,|\alpha_i\xi_n^k|^{1/n}\}=M(P(x^n)).$$

***Suggestion* 5**. If $P(x)=a_dx^d+\cdots+a_0=a_d\prod_i(x-\alpha_k)$, then $a_i/a_d=(-1)^{d-i}s_{d-i}(\alpha_k)$ where

$$s_j(\alpha_k)=\sum_{k_1<\cdots<k_j}\alpha_{k_1}\cdots\alpha_{k_j}$$

are the elementary symmetric polynomials. Observe that

$$|\alpha_{k_1}\cdots\alpha_{k_j}|\leq\prod_k\max\{1,|\alpha_k|\}.$$

It is then clear that

$$|a_i|=|a_ds_{d-i}(\alpha_k)|\leq\binom{d}{d-i}M(P)=\binom{d}{i}M(P).$$

***Suggestion* 6**. (a) The polynomial $5x^2-6x+5$ has $\frac{3+4i}{5}$ as a root.

(b) If a degree three integer polynomial has roots $\alpha_1,\alpha_2,\alpha_3$ with $|\alpha_1|=1$ and $\alpha_2=\overline{\alpha_1}$, then $\alpha_1\alpha_2\alpha_3=\alpha_1\overline{\alpha_1}\alpha_3=\alpha_3\in\mathbb{Z}$. Thus α_1 is a root of a monic integer quadratic which is impossible unless α_1 is a root of unity.

For degree 4, an example is given by

$$x^4+4x^3-2x^2+4x+1$$

whose root

$$\sqrt{2}-1+i\sqrt{2\sqrt{2}-2}$$

has absolute value 1.

***Suggestion* 7**. These values can be computed with `mahler.sage`

(a) 0.1623576120
(b) 0.1958888214
(c) 0.1823436598
(d) 0.1844998024

***Suggestion* 8**. Some of these values can be computed with `delta.sage`. However, for larger values it is better to use a recurrence of the Δ_n as proved in [Le33], section 8.

***Suggestion* 9**. We have

$$\left(\frac{y_1-1}{y_1}\cdots\frac{y_d-1}{y_d}\right)^{1/d}\leq\frac{\frac{y_1-1}{y_1}+\cdots+\frac{y_d-1}{y_d}}{d}$$

and

$$\frac{1}{(y_1\cdots y_d)^{1/d}} \le \frac{\frac{1}{y_1}+\cdots+\frac{1}{y_d}}{d}.$$

Summing both inequalities and multiplying by $(y_1\cdots y_d)^{1/d}$,

$$((y_1-1)\cdots(y_d-1))^{1/d}+1 \le (y_1\cdots y_d)^{1/d}.$$

***Suggestion* 10**. We see from Equation (1.1) and Exercise 9 that the condition for equality is $M(P) = \left(\frac{1+\sqrt{5}}{2}\right)^{d/2}$ that occurs when all the α_i with $|\alpha_i|>1$ are equal and all the α_i with $|\alpha_i|<1$ are equal. That means that $P(x) = ((x-\alpha)(x-\beta))^{d/2}$ with $|\alpha|>1$, $|\beta|<1$, and $M(P) = \left(\frac{1+\sqrt{5}}{2}\right)^{d/2}$. Thus, $\alpha = \frac{1+\sqrt{5}}{2}$, and we must have $\beta = \frac{1-\sqrt{5}}{2}$ and $P = (x^2-x-1)^{d/2}$.

***Suggestion* 11**. If we relax the condition $|P(0)|=1$, say that $|P(0)|=c$. Then $M(P) = \frac{c}{\prod_{|\alpha_i|<1}|\alpha_i|}$ and $E = c^2\left(M(P)^{2/d}-M(P)^{-2/d}\right)^d$. Thus, the conclusion is

$$M(P)^{2/d}-M(P)^{-2/d} \ge c^2.$$

Thus, $M(P) \ge \left(\frac{c^2\pm\sqrt{c^4+4}}{2}\right)^{d/2}$.

***Suggestion* 12**. Using exercise 5, we have (a)

$$L(P) = \sum_i |a_i| \le \sum_i \binom{d}{i} M(P) = 2^d M(P),$$

(b)

$$H(P) = \max_i |a_i| \le \max_i \binom{d}{i} M(P) \le 2^{d-1}M(P).$$

In the last step, we have used $\binom{d}{i} \le 2^{d-1}$ for $d\ge 1$ which can be proven by induction.

***Suggestion* 13**. (a) Write

$$P_{k_1,\dots,k_m}(x_{m+1},\dots,x_n) = \sum_{i_j} a_{k_1,\dots,k_m,i_{m+1},\dots,i_n} x_{m+1}^{i_{m+1}}\cdots x_n^{i_n}$$

Thus,

$$P(x_1,\dots,x_n) = \sum_{k_1=0}^{d_1} P_{k_1}(x_2,\dots,x_n)x_1^{k_1}$$

and

$$P_{k_1,\dots,k_{m-1}}(x_m,\dots,x_n) = \sum_{k_m=0}^{d_m} P_{k_1,\dots,k_m}(x_{m+1},\dots,x_n)x^{k_m}.$$

By Exercise 5, we get

$$|a_{k_1,\dots,k_n}| = M(P_{k_1,\dots,k_n}) \le \binom{d_m}{k_n} M(P_{k_1,\dots,k_{n-1}}) \le \binom{d_{m-1}}{k_{n-1}}\binom{d_m}{k_n} M(P_{k_1,\dots,k_{n-2}})\cdots \le \binom{d_1}{k_1}\cdots\binom{d_m}{k_n}M(P).$$

(b) The upper bound is proved in a similar way as Exercise 12(a). For the lower bound, just notice that

$$|P(e^{2\pi i\theta_1},\dots,e^{2\pi i\theta_n})| \le L(P).$$

(c) The upper bound is proved in a similar way as Exercise 12(b).

***Suggestion* 14**. Write $P(x_1,x_2) = a\left(1+\frac{Q(x_1,x_2)}{a}\right)$. Then $\mathrm{m}(P) = \log|a| + \mathrm{m}\left(1+\frac{Q(x_1,x_2)}{a}\right)$. Since $\left|\frac{Q(x_1,x_2)}{a}\right| < 1$, $\log\left(1+\frac{Q(x_1,x_2)}{a}\right)$ can be expanded as a Taylor series uniformly convergent in (x_1,x_2). The integral of this is zero, since the individual terms vanish. Taking the real part of the logarithm, $\mathrm{m}\left(1+\frac{Q(x_1,x_2)}{a}\right) = 0$.

***Suggestion* 15**. Using Euclidean algortithm, A can be written as a product of integer matrices which are of the following three types: 1) upper triangular with 1's in the diagonal, 2) lower triangular with 1's in the diagonal or 3) diagonal. It is easy to see that the Mahler measure is invariant by types 1) and 2) and for the type 3), we have, by setting $k_i\theta_i = \tau_i$,

$$\begin{aligned}
\int_0^1 \cdots \int_0^1 \log|P(e^{2\pi i k_1\theta_1}, \dots, e^{2\pi i k_n\theta_n})| d\theta_1 \dots d\theta_n &= \frac{1}{k_1\cdots k_n}\int_0^{k_1} \cdots \int_0^{k_n} \log|P(e^{2\pi i\tau_1}, \dots, e^{2\pi i\tau_n})| d\tau_1 \dots d\tau_n \\
&= \int_0^1 \cdots \int_0^1 \log|P(e^{2\pi i\tau_1}, \dots, e^{2\pi i\tau_n})| d\tau_1 \dots d\tau_n.
\end{aligned}$$

***Suggestion* 16**. This can be achieved by, for instance,

```
 sage: for n in range (1,100):
....:     mahler(x^n+x+1)
```

***Suggestion* 17**. By Jensen's formula, it is enough to evaluate

$$\begin{aligned}
I &= \int_0^1 \int_0^1 \log\max\left\{\left|1+e^{2\pi i\theta_1}\right|, \left|1+e^{2\pi i\theta_2}\right|\right\} d\theta_1 d\theta_2 \\
&= \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} \log\max\left\{\left|1+e^{it_1}\right|, \left|1+e^{it_2}\right|\right\} dt_1 dt_2 \\
&= \frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi} \log\max\left\{\left|1+e^{it_1}\right|, \left|1+e^{it_2}\right|\right\} dt_1 dt_2 \\
&= \frac{2}{\pi^2}\int_0^{\pi} \log\left|1+e^{it_2}\right| \int_{t_2}^{\pi} dt_1 dt_2 \\
&= \frac{2}{\pi^2}\int_0^{\pi} \log\left|1+e^{it_2}\right| (\pi - t_2) dt_2 \\
&= -\frac{2}{\pi^2}\int_0^{\pi} \log\left|1+e^{it_2}\right| t_2 dt_2.
\end{aligned}$$

In $0 \le t_2 < \pi$, we have

$$\begin{aligned}
I &= \mathrm{Re}\left(\frac{2}{\pi^2}\int_0^{\pi} t \sum_{n=1}^{\infty}\frac{(-1)^n e^{itn}}{n} dt\right) \\
&= \mathrm{Re}\left(\frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^{\pi} te^{itn} dt\right) \\
&= \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\,\mathrm{Re}\left(\frac{(-1)^n\pi}{in} + \frac{(-1)^n}{n^2} - \frac{1}{n^2}\right) \\
&= \frac{2}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{1}{n^3} - \frac{(-1)^n}{n^3}\right) \\
&= \frac{2}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{2}{n^3} - \frac{1+(-1)^n}{n^3}\right) \\
&= \frac{4}{\pi^2}\zeta(3) - \frac{4}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{(2m)^3} \\
&= \frac{7}{2\pi^2}\zeta(3).
\end{aligned}$$

***Suggestion* 18**. (a) We have

$$\mathrm{Li}_n(1) + \mathrm{Li}_n(-1) = \sum_{k=1}^{\infty}\frac{1+(-1)^k}{k^n} = 2\sum_{k=1}^{\infty}\frac{1}{(2k)^n} = \frac{1}{2^{n-1}}\mathrm{Li}_n(1)$$

Therefore,

$$\mathrm{Li}_n(-1) = \left(\frac{1}{2^{n-1}} - 1\right)\mathrm{Li}_n(1) = \left(\frac{1}{2^{n-1}} - 1\right)\zeta(n).$$

(b) In this case we have

$$\begin{aligned}
\mathrm{Li}_n(e^{2\pi i/3}) - \mathrm{Li}_n(e^{-2\pi i/3}) &= \sum_{k=1}^{\infty} \frac{e^{2k\pi i/3} - e^{-2k\pi i/3}}{k^n} \\
&= \sum_{k=1}^{\infty} \frac{2\sin\left(\frac{2k\pi}{3}\right)}{k^n} \\
&= \sqrt{3}\sum_{k=1}^{\infty} \frac{\chi_{-3}(k)}{k^n} \\
&= \sqrt{3}L(\chi_{-3}, n).
\end{aligned}$$

***Suggestion* 20**. We have

$$\zeta(n_1, \dots, n_m) = (-1)^m I_{n_1,\dots,n_m}(1 : \dots : 1).$$

Making the change of variable $t \to 1-t$ for each variable in the integral, we obtain

$$\begin{aligned}
&(-1)^m \int_0^1 \underbrace{\frac{dt}{t-1} \circ \frac{dt}{t} \circ \dots \circ \frac{dt}{t}}_{n_1} \circ \dots \circ \underbrace{\frac{dt}{t-1} \circ \frac{dt}{t} \circ \dots \circ \frac{dt}{t}}_{n_m} \\
&= (-1)^{m+\sum n_k} \int_0^1 \underbrace{\frac{dt}{t-1} \circ \dots \circ \frac{dt}{t-1} \circ \frac{dt}{t}}_{n_m} \circ \dots \circ \underbrace{\frac{dt}{t-1} \circ \dots \circ \frac{dt}{t-1} \circ \frac{dt}{t}}_{n_1} \\
&= (-1)^{m+\sum n_k} I_{\underbrace{1,\dots,1}_{n_m-2},2,\dots,\underbrace{1,\dots,1}_{n_1-2},2}(1 : \dots : 1) \\
&= \zeta(\underbrace{1,\dots,1}_{n_m-2}, 2, \dots, \underbrace{1,\dots,1}_{n_1-2}, 2).
\end{aligned}$$

***Suggestion* 21**. First notice that this Mahler measure depends only on $|a|$ (since y and z can absorb number of absolute value 1 in the integration). For $|a| < 1$ this proof is done exactly in the same way as Exercise 17. For $|a| > 1$, we have,

$$\mathrm{m}(1 + x + ay + az) = \log|a| + \mathrm{m}\left(\frac{1+x}{a} + y + z\right) = \log|a| + \mathrm{m}\left(1 + x + \frac{y+z}{a}\right)$$

since we can exchange the variables x, y, z and the constant term in the integral by symmetry. Then apply the formula for $|a| < 1$.

***Suggestion* 25**. Observe that

$$\begin{aligned}
&\left(\int_0^1 \frac{dt}{t - x^{-1}}\right)^j \left(\int_0^1 \frac{dt}{t-x}\right)^{k-j} \\
= \quad & j!(k-j)! \int_0^1 \underbrace{\frac{dt}{t-x^{-1}} \circ \dots \circ \frac{dt}{t-x^{-1}}}_{j} \int_0^1 \underbrace{\frac{dt}{t-x} \circ \dots \circ \frac{dt}{t-x}}_{k-j}.
\end{aligned}$$

Combining the previous equalities gives

$$\begin{aligned}
&\mathrm{m}_k(1-x) = \frac{1}{2\pi i}\int_{|x|=1} \log^k|1-x| \frac{dx}{x} \\
= \quad & \frac{k!}{2^k}\sum_{j=0}^{k} \frac{1}{2\pi i}\int_{|x|=1}\int_0^1 \underbrace{\frac{dt}{t-x^{-1}} \circ \dots \circ \frac{dt}{t-x^{-1}}}_{j} \int_0^1 \underbrace{\frac{dt}{t-x} \circ \dots \circ \frac{dt}{t-x}}_{k-j} \frac{dx}{x}.
\end{aligned}$$

If we now set $s = xt$ in the first j-fold integral and $s = \frac{t}{x}$ in the second $(k-j)$-fold integral, the above becomes

$$\frac{k!}{2^k}\sum_{j=0}^{k}\frac{1}{2\pi i}\int_{|x|=1}\int_0^x \frac{ds}{s-1}\circ\cdots\circ\frac{ds}{s-1}\int_0^{x^{-1}}\frac{ds}{s-1}\circ\cdots\circ\frac{ds}{s-1}\frac{dx}{x}.$$

We proceed to compute the integrals in terms of multiple polylogarithms:

$$\begin{aligned}
\mathrm{m}_k(1-x) &= \frac{(-1)^k k!}{2^k}\sum_{j=0}^{k}\frac{1}{2\pi i}\int_{|x|=1}\left(\sum_{0<l_1<\cdots<l_j<\infty}\sum_{0<m_1<\cdots<m_{k-j}<\infty}\frac{x^{l_j-m_{k-j}}}{l_1\ldots l_j m_1\ldots m_{k-j}}\right)\frac{dx}{x}\\
&= \frac{(-1)^k k!}{2^k}\sum_{j=1}^{k-1}\sum_{0<l_1<\cdots<l_{j-1}<u<\infty,\,0<m_1<\cdots<m_{k-j-1}<u<\infty}\frac{1}{l_1\ldots l_{j-1}m_1\ldots m_{k-j-1}u^2}.
\end{aligned}$$

Now we need to analyze each term of the form

$$\sum_{0<l_1<\cdots<l_{j-1}<u<\infty,\,0<m_1<\cdots<m_{k-j-1}<u<\infty}\frac{1}{l_1\ldots l_{j-1}m_1\ldots m_{k-j-1}u^2}. \tag{8.1}$$

For an h-tuple $a_1,\ldots,a_h$ such that $a_1+\cdots+a_h = k-2h$, we set

$$d_{a_1,\ldots,a_h} = \sum_{e_1+\cdots+e_h=j-h}\binom{a_1}{e_1}\cdots\binom{a_h}{e_h} = \binom{a_1+\cdots+a_h}{e_1+\cdots+e_h} = \binom{k-2h}{j-h}.$$

Then the term (8.1) is equal to

$$\sum_{h=1}^{\min\{j-1,k-j-1\}} d_{a_1,\ldots,a_h}\zeta(\{1\}_{a_1},2,\ldots,\{1\}_{a_h},2).$$

Note that each term $\zeta(\{1\}_{a_1},2,\ldots,\{1\}_{a_h},2)$ comes from choosing $h-1$ of the l's and $h-1$ of the m's and making them equal in pairs. Once this process has been done, one can choose the way the other l's and m's are ordered. All these choices give rise to the coefficients $d_{a_1,\ldots,a_h}$.

The total sum is given by

$$\mathrm{m}_k(1-x) = \sum_{h=1}^{k-1} c_{a_1,\ldots,a_h}\zeta(\{1\}_{a_1},2,\ldots,\{1\}_{a_h},2),$$

where

$$c_{a_1,\ldots,a_h} = \frac{(-1)^k k!}{2^k}\sum_{j=1}^{k-1}\binom{k-2h}{j-h} = \frac{(-1)^k k!}{2^k}2^{k-2h} = \frac{(-1)^k k!}{2^{2h}}.$$

On the other hand,

$$\zeta(\{1\}_{a_1},2,\ldots,\{1\}_{a_h},2) = \zeta(a_h+2,\ldots,a_1+2).$$

Thus, the total sum is

$$\mathrm{m}_k(1-x) = \sum_{b_1+\cdots+b_h=k,\,b_i\geq 2}\frac{(-1)^k k!}{2^{2h}}\zeta(b_1,\ldots,b_h).$$

***Suggestion* 26**. First assume that $|\alpha|,|\beta|\leq 1$. Then the proof is similar to Theorem 5.5, the final step being

$$\mathrm{m}(1-\alpha x, 1-\beta x) = \frac{1}{2}\sum_{k=1}^{\infty}\frac{\cos(2\pi k\tau)|\alpha\beta|^k}{k^2},$$

where $e^{2\pi i\tau} = \frac{\beta|\alpha|}{|\beta|\alpha}$.

If $|\alpha| > 1$ and $|\beta| \le 1$, we have that $\mathrm{m}(1-\beta x) = 0$ and

$$\begin{aligned}\mathrm{m}(1-\alpha x, 1-\beta x) &= \mathrm{m}(\alpha, 1-\beta x) + \mathrm{m}(\alpha^{-1} - x, 1-\beta x)\\ &= \log|\alpha|\mathrm{m}(1-\beta x) + \mathrm{m}(1-\alpha^{-1}x, 1-\beta x)\\ &= \mathrm{m}(1-\alpha^{-1}x, 1-\beta x),\end{aligned}$$

and we are in the first case.

For $|\alpha| > 1$ and $|\beta| > 1$, the previous identity becomes

$$\mathrm{m}(1-\alpha x, 1-\beta x) = \log|\alpha|\log|\beta| + \mathrm{m}(1-\alpha^{-1}x, 1-\beta x)$$

and we apply the formula for the case $|\beta| > 1$ and $|\alpha^{-1}| < 1$.

***Suggestion* 27**. Following a similar proof to Theorem 5.5, it is not hard to see that we are reduced to compute

$$I = \int_0^1 \cos(2\pi k\theta)\cos(2\pi l(\theta+\alpha))\cos(2\pi m(\theta+\beta))d\theta.$$

But this equals

$$\begin{aligned}I &= \frac{1}{2}\int_0^1 \cos(2\pi k\theta)(\cos(2\pi((l+m)\theta + l\alpha + m\beta)) + \cos(2\pi((l-m)\theta + l\alpha - m\beta)))d\theta\\ &= \begin{cases} \frac{1}{4}\cos(2\pi(l\alpha+m\beta)) & \text{if } l+m=k,\\ \frac{1}{4}\cos(2\pi(l\alpha-m\beta)) & \text{if } l-m=k,\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

***Suggestion* 28**. (a) For any positive integer h, let f and g be functions such that

$$\frac{1}{2\pi i}\int_{|x|=1} |f|^h \frac{dx}{x} < \infty \qquad \text{and} \qquad \frac{1}{2\pi i}\int_{|x|=1} |g|^{h/(h-1)}\frac{dx}{x} < \infty.$$

Then, by Hölder's inequality, we get that

$$\left(\frac{1}{2\pi i}\int_{|x|=1}|fg|\frac{dx}{x}\right)^h \le \left(\frac{1}{2\pi i}\int_{|x|=1}|f|^h\frac{dx}{x}\right)\left(\frac{1}{2\pi i}\int_{|x|=1}|g|^{h/(h-1)}\frac{dx}{x}\right)^{h-1}. \tag{8.2}$$

In particular, taking $f(x) = \log^2|P(x)|$ and $g(x) = 1$, we get that

$$\mathrm{m}_2(P)^h \le \mathrm{m}_{2h}(P).$$

(b) On the other hand, by taking $f(x) = \log|P(x)|$ and $g(x) = 1$, and taking $2h$ instead of h in (8.2) we get that

$$\mathrm{m}(P)^{2h} \le \mathrm{m}_{2h}(P).$$

***Suggestion* 29**. These values can by computed with `mahler2.sage`
(a) 1.7447964556
(b) 1.2863292447
(c) 1.3885013172
(d) 1.3845721865

***Suggestion* 30**. We set $a=1$ in Theorem 4.4:

$$\begin{aligned}
\mathrm{m}((1+x)+(1-x)y) &= -\frac{i}{\pi}\left(\mathrm{Li}_2\left(i\right)-\mathrm{Li}_2\left(-i\right)\right)\\
&= -\frac{i}{\pi}\sum_{n=1}^{\infty}\left(\frac{i^n}{n^2}-\frac{(-i)^n}{n^2}\right)\\
&= -\frac{1}{\pi}\sum_{k=1}^{\infty}\left(\frac{i^{2k}}{(2k-1)^2}+\frac{(-i)^{2k}}{(2k-1)^2}\right)\\
&= -\frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k-1)^2}\\
&= \frac{2}{\pi}\sum_{n=1}\frac{\chi_{-4}(n)}{n^2},
\end{aligned}$$

where

$$\chi_{-4}(n)=\left\{\begin{array}{rll} 1 & \text{if} & n\equiv 1 \mod 4,\\ -1 & \text{if} & n\equiv -1 \mod 4,\\ 0 & \text{if} & n\equiv 0 \mod 2.\end{array}\right.$$

***Suggestion* 31**. (a) and (c) are obvious. For (b), we may use the Taylor expansions in λ;

$$(1+\lambda P)^s=\sum_{k=0}^{\infty}\binom{s}{k}\lambda^kP^k,\quad \log(1+\lambda P)=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\lambda^kP^k.$$

In particular, we may write

$$Z(s,1+\lambda P)=\sum_{k=0}^{\infty}\mathrm{m}_k(1+\lambda P)\frac{s^k}{k!}=\sum_{k=0}^{\infty}Z(k,P)\lambda^k\frac{s(s-1)\cdots(s-k+1)}{k!}.$$

In other words, the coefficients with respect to the monomial basis are the k-logarithmic Mahler measures $\mathrm{m}_k(1+\lambda P)$, while the coefficients with respect to the shifted monomial basis are (the special values of) zeta Mahler measures $Z(k,P)\lambda^k$.

Combining these observations, we obtain the three equalities.

***Suggestion* 32**. We have

$$\begin{aligned}
\mathrm{m}_2\left(\frac{1+x+y}{1+x+z}\right) &= \mathrm{m}_2(1+x+y)-2\mathrm{m}(1+x+y,1+x+z)+\mathrm{m}_2(1+x+z)\\
&= 2\left(\frac{3}{\pi}\mathrm{Ls}_3\left(\frac{2\pi}{3}\right)+\frac{\pi^2}{4}\right)-2\frac{\pi^2}{54}\\
&= \frac{6}{\pi}\mathrm{Ls}_3\left(\frac{2\pi}{3}\right)+\frac{25\pi^2}{54}.
\end{aligned}$$

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