Regulators and computations of Mahler measures

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Mahler measure of multivariable polynomials

$P \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, the (logarithmic) *Mahler measure* is :

$$\begin{aligned} m(P) &= \int_0^1 \ldots \int_0^1 \log |P(\mathrm{e}^{2\pi \mathrm{i}\theta_1}, \ldots, \mathrm{e}^{2\pi \mathrm{i}\theta_n})| \mathrm{d}\theta_1 \ldots \mathrm{d}\theta_n \\ &= \frac{1}{(2\pi \mathrm{i})^n} \int_{\mathbb{T}^n} \log |P(x_1, \ldots, x_n)| \frac{\mathrm{d}x_1}{x_1} \ldots \frac{\mathrm{d}x_n}{x_n} \end{aligned}$$

Smyth (1981)

$$m(1+x+y) = \frac{3\sqrt{3}}{4\pi}\mathrm{L}(\chi_{-3},2) = \mathrm{L}'(\chi_{-3},-1)$$

$$\mathrm{L}(\chi_{-3},s) = \sum_{n=1}^{\infty}\frac{\chi_{-3}(n)}{n^s} \qquad \chi_{-3}(n) = \begin{cases} 1 & n \equiv 1 \bmod 3 \\ -1 & n \equiv -1 \bmod 3 \\ 0 & n \equiv 0 \bmod 3 \end{cases}$$

Polylogarithms

The *kth polylogarithm* is

$$\mathrm{Li}_k(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^k} \qquad x \in \mathbb{C}, \quad |x| < 1$$

It has an analytic continuation to $\mathbb{C} \setminus [1, \infty)$.

Zagier:

$$\widehat{\mathcal{L}_k}(x) := \widehat{\mathrm{Re}_k} \left(\sum_{j=0}^{k-1} \frac{2^j B_j}{j!} (\log |x|)^j \mathrm{Li}_{k-j}(x) \right)$$

B_j is jth Bernoulli number

$\widehat{\mathrm{Re}_k} = \mathrm{Re}$ or $\mathrm{i}\,\mathrm{Im}$ if k is odd or even.

One-valued, real analytic in $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$, continuous in $\mathbb{P}^1(\mathbb{C})$.

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One-valued, real analytic in $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$, continuous in $\mathbb{P}^1(\mathbb{C})$.

$\widehat{\mathcal{L}_k}$ satisfies lots of functional equations

$$\widehat{\mathcal{L}_k}\left(\frac{1}{x}\right) = (-1)^{k-1}\widehat{\mathcal{L}_k}(x) \qquad \widehat{\mathcal{L}_k}(\bar{x}) = (-1)^{k-1}\widehat{\mathcal{L}_k}(x)$$

Bloch–Wigner dilogarithm ($k = 2$)

$$D(x) := \mathrm{Im}(\mathrm{Li}_2(x)) + \arg(1-x)\log|x|$$

Five-term relation

$$D(x) + D(1-xy) + D(y) + D\left(\frac{1-y}{1-xy}\right) + D\left(\frac{1-x}{1-xy}\right) = 0$$

The relation with regulators

Deninger (1997)

$$m(P) = m(P^*) + \frac{1}{(-2i\pi)^{n-1}} \int_\Gamma \eta_n(n)(x_1, \dots, x_n)$$

where

$$\Gamma = \{P(x_1, \dots, x_n) = 0\} \cap \{|x_1| = \dots = |x_{n-1}| = 1, |x_n| \geq 1\}$$

$\eta_n(n)(x_1, \dots, x_n)$ is a $\mathbb{R}(n-1)$-valued smooth $n-1$-form in $X(\mathbb{C})$.

Philosophy of Beilinson's conjectures

Global information from local information through L-functions

- Arithmetic-geometric object X (for instance, $X = \mathcal{O}_F$, F a number field)
- L-function ($\mathrm{L}_F = \zeta_F$)
- Finitely-generated abelian group K ($K = \mathcal{O}_F^*$)
- Regulator map $\mathrm{reg} : K \to \mathbb{R}$ ($\mathrm{reg} = \log|\cdot|$)

$$(K \,\mathrm{rank}\, 1) \qquad \mathrm{L}'_X(0) \sim_{\mathbb{Q}^*} \mathrm{reg}(\xi)$$

(Dirichlet class number formula, for F real quadratic, $\zeta'_F(0) \sim_{\mathbb{Q}^*} \log|\epsilon|$, $\epsilon \in \mathcal{O}_F^*$)

Example:

$$\eta_2(2)(x,y) = \log|x|\,\mathrm{d}i\arg y - \log|y|\,\mathrm{d}i\arg x$$

Smyth (1981)

$$m(1+x+y) = \frac{1}{(2\pi\mathrm{i})^2}\int_{\mathbb{T}^2}\log|1+x+y|\frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y}$$

by Jensen's equality:

$$= \frac{1}{2\pi\mathrm{i}}\int_{\mathbb{T}^1}\log^+|1+x|\frac{\mathrm{d}x}{x}$$

$$= \frac{1}{2\pi\mathrm{i}}\int_{\Gamma}\log|y|\frac{\mathrm{d}x}{x} = -\frac{1}{2\pi\mathrm{i}}\int_{\Gamma}\eta_2(2)(x,y)$$

where

$$\Gamma = \{1+x+y=0\}\cap\{|x|=1, |y|\geq 1\}$$

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where

$$\Gamma = \{1+x+y=0\}\cap\{|x|=1, |y|\geq 1\}$$

but

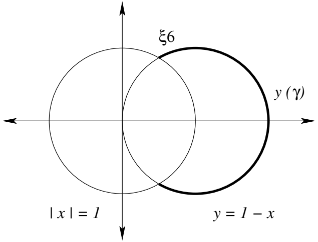
$$\eta_2(2)(x, 1-x) = \mathrm{d}D(x)$$

and

$$m(1+x+y) = -\frac{1}{2\pi \mathrm{i}} \int_\Gamma \eta_2(2)(x,y) \qquad 1+x+y=0$$

Use Stokes Theorem:

$$m(P) = -\frac{1}{2\pi} D(\partial\gamma)$$



$$2\pi m(x+y+1) = D(\xi_6) - D(\bar{\xi_6}) = \frac{3\sqrt{3}}{2} \mathrm{L}(\chi_{-3}, 2)$$

Properties of $\eta_n(n)(x_1, \ldots, x_n)$

- Multiplicative in each variable, anti-symmetric.
  $\eta_n(n)$ is a function on $\bigwedge^n (\mathbb{C}(X)^*)_{\mathbb{Q}}$

- $\mathrm{d}\eta_n(n)(x_1, \ldots, x_n) = \widehat{\mathrm{Re}_n}\left(\frac{\mathrm{d}x_1}{x_1} \wedge \cdots \wedge \frac{\mathrm{d}x_n}{x_n}\right)$

- $\eta_n(n)(x, 1-x, x_1, \ldots, x_{n-2}) = \mathrm{d}\eta_{n-1}(n)(x, x_1, \ldots, x_{n-2})$

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Examples

$$\eta_2(2)(x, 1-x) = \mathrm{d}\widehat{D}(x)$$

$$\eta_3(3)(x,y,z) = \log|x|\left(\frac{1}{3}\mathrm{d}\log|y| \wedge \mathrm{d}\log|z| + \mathrm{d}i\arg y \wedge \mathrm{d}i\arg z\right)$$

$$+\log|y|\left(\frac{1}{3}\mathrm{d}\log|z| \wedge \mathrm{d}\log|x| + \mathrm{d}i\arg z \wedge \mathrm{d}i\arg x\right)$$

$$+\log|z|\left(\frac{1}{3}\mathrm{d}\log|x| \wedge \mathrm{d}\log|y| + \mathrm{d}i\arg x \wedge \mathrm{d}i\arg y\right)$$

$$\eta_3(3)(x, 1-x, y) = \mathrm{d}\eta_3(2)(x,y)$$

$$\eta_3(2)(x,y)$$

$$= \widehat{D}(x)\mathrm{d}i\arg y + \frac{1}{3}\log|y|(\log|1-x|\mathrm{d}\log|x| - \log|x|\mathrm{d}\log|1-x|)$$

Examples

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First variable in $\eta_n(n-1)$ behaves like the five-term relation

$$[x]+[y]+[1-xy]+\left[\frac{1-x}{1-xy}\right]+\left[\frac{1-x}{1-xy}\right]$$

Now

$$\eta_n(n-1)(x,x,x_1,\ldots,x_{n-3}) = \mathrm{d}\eta_n(n-2)(x,x_1,\ldots,x_{n-3})$$

First variable in $\eta_n(n-2)$ behaves like rational functional equations of $\mathcal{L}_3$.

. . .

$$\eta_n(2)(x,x) = \mathrm{d}\eta_n(1)(x)$$

and

$$\eta_n(1)(x) = \widehat{\mathcal{L}_n}(x)$$

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Example in three variables

Smyth (1981)

$$m(1-x+(1-y)z) = \frac{7}{2\pi^2}\zeta(3)$$

$$m(P) = m(1-y) - \frac{1}{(2\pi)^2}\int_\Gamma \eta_3(3)(x,y,z).$$

$$x \wedge y \wedge z = - \; x \wedge (1-x) \wedge y - \; y \wedge (1-y) \wedge x,$$

$$x \wedge y \wedge z = - \; x \wedge (1-x) \wedge y - \; y \wedge (1-y) \wedge x,$$

in other words,

$$\eta_3(3)(x,y,z) = -\eta_3(3)(x,1-x,y) - \eta_3(3)(y,1-y,x).$$

We have

$$m((1-x)+(1-y)z) = \frac{1}{4\pi^2}\int_\gamma \eta_3(2)(x,y) + \eta_3(2)(y,x).$$

$$\Gamma = \{P(x,y,z)=0\} \cap \{|x|=|y|=1, |z|\geq 1\}$$

$$\partial\Gamma = \{P(x,y,z)=0\} \cap \{|x|=|y|=|z|=1\}$$

Maillot: $P \in \mathbb{R}[x,y,z]$,

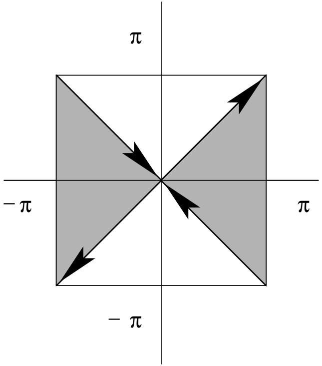
$$\gamma = \{P(x,y,z) = P(x^{-1},y^{-1},z^{-1})=0\} \cap \{|x|=|y|=1\}.$$

$$C = \{P(x,y,z) = P(x^{-1},y^{-1},z^{-1})=0\}$$

$$\frac{(1-x)(1-x^{-1})}{(1-y)(1-y^{-1})} = 1$$

$$C = \{x=y\} \cup \{xy=1\}$$

$$-\{x\}_2 \otimes y - \{y\}_2 \otimes x = \pm 2\{x\}_2 \otimes x.$$



$$m((1-x)+(1-y)z) = \frac{1}{4\pi^2} 8(\mathcal{L}_3(1) - \mathcal{L}_3(-1)) = \frac{7}{2\pi^2}\zeta(3).$$

Other examples

Boyd & L. (2005)

$$m(x^2+1+(x+1)y+(x-1)z) = \frac{\mathrm{L}(\chi_{-4},2)}{\pi} + \frac{21}{8\pi^2}\zeta(3)$$

$$m(x^2+x+1+(x+1)y+z) = \frac{\sqrt{3}}{4\pi}\mathrm{L}(\chi_{-3},2) + \frac{19}{6\pi^2}\zeta(3)$$

An example in four variables

L.(2003)

$$\pi^3 m\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)(1+y)z\right) = 2\pi^2 \mathrm{L}(\chi_{-4},2) + 8\sum_{0\le j<k}\frac{(-1)^{j+k+1}}{(2j+1)^3 k}$$

(2005)

$$= 24\mathrm{L}(\chi_{-4},4)$$

In general, for m odd,

$$\sum_{0\le j<k}\frac{(-1)^{j+k+1}}{(2j+1)^m k}$$

$$= m\mathrm{L}(\chi_{-4},m+1) + \sum_{h=1}^{\frac{m-1}{2}}\frac{(-1)^h\pi^{2h}(2^{2h}-1)}{(2h)!}B_{2h}\mathrm{L}(\chi_{-4},m-2h+1)$$

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Generalized Mahler measure

Gon & Oyanagi (2004)

For $f_1, \dots, f_r \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$,

$$m(f_1, \dots, f_r) = \frac{1}{(2\pi \mathrm{i})^n} \int_{\mathbb{T}^n} \max\{\log |f_1|, \dots, \log |f_r|\} \frac{\mathrm{d}x_1}{x_1} \dots \frac{\mathrm{d}x_n}{x_n}$$

Note

$$m(f_1, f_2) = m(f_1 + z f_2)$$

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Examples

The particular case when $f_j = P(x_j)$ for some $P \in \mathbb{C}[x]$.
Gon & Oyanagi (2004)

$$m(1-x_1, \dots, 1-x_n) = \sum_{j=1}^{[\frac{n}{2}]} c_{j,n} \frac{\zeta(2j+1)}{\pi^{2j}}$$

$$m(1-x_1, 1-x_2) = \frac{7}{2\pi^2}\zeta(3)$$

$$m(1-x_1, 1-x_2, 1-x_3) = \frac{9}{2\pi^2}\zeta(3)$$

$$m(1-x_1, 1-x_2, 1-x_3, 1-x_4) = -\frac{93}{2\pi^4}\zeta(5) + \frac{9}{\pi^2}\zeta(3)$$

Can be also computed using regulators.

$|P(x)|$ is montononous when $0 \leq \arg x \leq \pi$.

In this case, $|P(x)| = 2 \left| \sin \frac{\arg x}{2} \right|$.

$$m(P(x_1), \ldots, P(x_n)) = \frac{n!}{(\pi \mathrm{i})^n} \int_{0 \leq \arg x_n \leq \cdots \leq \arg x_1 \leq \pi} \eta(P(x_1), x_1, \ldots, x_n)$$

L. (2005)

$$m\left(\frac{1-x_1}{1+x_1},\dots,\frac{1-x_n}{1+x_n}\right)=\sum_{j=1}^{\left[\frac{n}{2}\right]} c'_{j,n}\frac{\zeta(2j+1)}{\pi^{2j}}$$

$$m\left(\frac{1-x_1}{1+x_1},\frac{1-x_2}{1+x_2}\right)=\frac{7}{\pi^2}\zeta(3)$$

$$m\left(\frac{1-x_1}{1+x_1},\dots,\frac{1-x_3}{1+x_3}\right)=\frac{21}{2\pi^2}\zeta(3)$$

$$m\left(\frac{1-x_1}{1+x_1},\dots,\frac{1-x_4}{1+x_4}\right)=-\frac{93}{\pi^4}\zeta(5)+\frac{21}{\pi^2}\zeta(3)$$

$m(1+x_1-x_1^{-1},\dots,1+x_n-x_n^{-1}) =$ combination of polylogarithms.

$$m(1+x_1-x_1^{-1}) = -\log(\varphi),$$

$$m(1+x_1-x_1^{-1}, 1+x_2-x_2^{-1})$$
$$= \frac{1}{\pi^2}\,\mathrm{Re}(\mathrm{Li}_3(\varphi^2) - \mathrm{Li}_3(-\varphi^2) + \mathrm{Li}_3(\varphi^{-2}) - \mathrm{Li}_3(-\varphi^{-2}))$$

for $\varphi = \frac{-1+\sqrt{5}}{2}$.

A result about generalized Mahler measure

Let $P \in \mathbb{C}(x_1, \dots, x_n)$ and let $f_i = P(x_{i,1}, \dots, x_{i,n})$ for $i = 1, \dots r$. Then

$$\lim_{r \to \infty} m(f_1, \dots, f_r) = \log ||P||_\infty$$

where $|| \cdot ||_\infty$ stands for the sup norm on $\mathbb{T}^n$.

Functional equations for Mahler measures of genus-one curves

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The measures of a family of genus-one curves

$$m(k) := m\left(x + \frac{1}{x} + y + \frac{1}{y} + k\right)$$

Boyd 1998

$$m(k) \stackrel{?}{=} \frac{\mathrm{L}'(E_k, 0)}{s_k} \quad k \in \mathbb{N} \neq 0, 4$$

E_k determined by $x + \frac{1}{x} + y + \frac{1}{y} + k = 0$.

Deninger 1997

L-functions $\leftarrow$ Beilinson's conjectures
Kronecker-Eisenstein series for $k = 1$

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Rodriguez-Villegas 1997

$k = 4\sqrt{2}$ (CM case)

$$m\left(4\sqrt{2}\right) = m\left(x + \frac{1}{x} + y + \frac{1}{y} + 4\sqrt{2}\right) = \mathrm{L}'(E_{4\sqrt{2}}, 0)$$

$k = 3\sqrt{2}$ (modular curve $X_0(24)$)

$$m\left(3\sqrt{2}\right) = m\left(x + \frac{1}{x} + y + \frac{1}{y} + 3\sqrt{2}\right) = q\mathrm{L}'(E_{3\sqrt{2}}, 0)$$

$$q \in \mathbb{Q}^*, \quad q \stackrel{?}{=} \frac{5}{2}$$

Theorem

(Rodriguez-Villegas) $E_k \sim$ modular elliptic surface assoc $\Gamma_0(4)$.

$$m(k) = \mathrm{Re}\left(\frac{16y_\mu}{\pi^2}\sum_{m,n}{}' \frac{\chi_{-4}(m)}{(m+n4\mu)^2(m+n4\bar{\mu})}\right)$$

$$= \mathrm{Re}\left(-\pi \mathrm{i}\mu + 2\sum_{n=1}^{\infty}\sum_{d|n}\chi_{-4}(d)d^2\frac{q^n}{n}\right)$$

where $j(E_k) = j\left(-\frac{1}{4\mu}\right)$

$$q = \mathrm{e}^{2\pi \mathrm{i}\mu} = q\left(\frac{16}{k^2}\right) = \exp\left(-\pi\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1,1-\frac{16}{k^2}\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1,\frac{16}{k^2}\right)}\right)$$

and y_μ is the imaginary part of μ.

Theorem

(Kurokawa & Ochiai 2005)
For $h \in \mathbb{R}^*$,

$$m(4h^2) + m\left(\frac{4}{h^2}\right) = 2m\left(2\left(h + \frac{1}{h}\right)\right).$$

(L. & Rogers 2007)
For $|h| < 1$, $h \neq 0$,

$$m\left(2\left(h + \frac{1}{h}\right)\right) + m\left(2\left(\mathrm{i}h + \frac{1}{\mathrm{i}h}\right)\right) = m\left(\frac{4}{h^2}\right).$$

$$m\left(2\left(h + \frac{1}{h}\right)\right) - m\left(2\left(\mathrm{i}h + \frac{1}{\mathrm{i}h}\right)\right) = m\left(4h^2\right).$$

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Corollary

$$m(8) = 4m(2) = \frac{8}{5}m(3\sqrt{2})$$

$$m\left(3\sqrt{2}\right) = q\mathrm{L}'(E_{3\sqrt{2}}, 0)$$

$$q \in \mathbb{Q}^*, \quad q \stackrel{?}{=} \frac{5}{2}$$

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The elliptic regulator

F field. Matsumoto Theorem:

$$K_2(F) = \langle \{a, b\}, a, b \in F \rangle \,/\, \langle \text{bilinear}, \{a, 1-a\} \rangle$$

$K_2(E) \otimes \mathbb{Q}$ subgroup of $K_2(\mathbb{Q}(E)) \otimes \mathbb{Q}$ determined by kernels of tame symbols.

$x, y \in \mathbb{C}(E)$

$$\eta(x, y) := \log|x| \, \mathrm{d} \arg y - \log|y| \, \mathrm{d} \arg x$$

1-form on $E(\mathbb{C}) \setminus S$
for any loop $\gamma \in E(\mathbb{C}) \setminus S$

$$(\gamma, \eta(x, y)) = \frac{1}{2\pi} \int_\gamma \eta(x, y)$$

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The regulator map (Beilinson, Bloch):

$$r : K_2(E) \otimes \mathbb{Q} \to H^1(E, \mathbb{R})$$

$$\{x, y\} \to \left\{ \gamma \to \int_\gamma \eta(x, y) \right\}$$

for $\gamma \in H_1(E, \mathbb{Z})$. ($H^1(E, \mathbb{R})$ dual of $H_1(E, \mathbb{Z})$)

Follows from $\eta(x, 1-x) = \mathrm{d}D(x)$,

$$D(x) = \mathrm{Im}(\mathrm{Li}_2(x)) + \arg(1-x) \log |x|$$

is the Bloch-Wigner dilogarithm

The relation with Mahler measures

Deninger

$$m(k) \sim_{\mathbb{Z}} \frac{1}{2\pi} r(\{x, y\})(\gamma)$$

In the example,

$$yP_k(x, y) = (y - y_{(1)}(x))(y - y_{(2)}(x)),$$

$$m(k) = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{T}^1} (\log^+ |y_{(1)}(x)| + \log^+ |y_{(2)}(x)|) \frac{\mathrm{d}x}{x}.$$

By Jensen's formula respect to y.

$$m(k) = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{T}^1} \log |y| \frac{\mathrm{d}x}{x} = -\frac{1}{2\pi} \int_{\mathbb{T}^1} \eta(x, y),$$

$\mathbb{T}^1 \in H_1(E, \mathbb{Z})$.

Computing the regulator

$$E(\mathbb{C}) \cong \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z} \cong \mathbb{C}^*/q^{\mathbb{Z}}$$

$z \bmod \Lambda = \mathbb{Z} + \tau\mathbb{Z}$ is identified with $\mathrm{e}^{2\mathrm{i}\pi z}$.
Bloch regulator function

$$R_\tau\left(\mathrm{e}^{2\pi\mathrm{i}(a+b\tau)}\right) = \frac{y_\tau^2}{\pi} \sum_{m,n\in\mathbb{Z}}{}' \frac{\mathrm{e}^{2\pi\mathrm{i}(bn-am)}}{(m\tau+n)^2(m\bar{\tau}+n)}$$

y_τ is the imaginary part of τ.
Elliptic dilogarithm

$$D_\tau(z) := \sum_{n\in\mathbb{Z}} D(zq^n)$$

Regulator function given by

$$R_\tau = D_\tau - \mathrm{i}J_\tau$$

$\mathbb{Z}[E(\mathbb{C})]^- = \mathbb{Z}[E(\mathbb{C})]/\sim \quad [-P] \sim -[P]$.
R_τ is an odd function,

$$\mathbb{Z}[E(\mathbb{C})]^- \to \mathbb{C}.$$

$$(x) = \sum m_i(a_i), \qquad (y) = \sum n_j(b_j).$$

$$\mathbb{C}(E)^* \otimes \mathbb{C}(E)^* \to \mathbb{Z}[E(\mathbb{C})]^-$$
$$(x) \diamond (y) = \sum m_i n_j (a_i - b_j).$$

Proposition

$E/\mathbb{R}$ elliptic curve, x, y are non-constant functions in $\mathbb{C}(E)$ with trivial tame symbols, $\omega \in \Omega^1$

$$-r(\{x,y\}) = -\int_\gamma \eta(x,y) = \mathrm{Im}\left(\frac{\Omega}{y_\tau \Omega_0} R_\tau\left((x) \diamond (y)\right)\right)$$

where Ω_0 is the real period and $\Omega = \int_\gamma \omega$.

Use results of Beilinson, Bloch, Deninger

Idea of Proof

$$x + \frac{1}{x} + y + \frac{1}{y} + k = 0$$

Weierstrass form:

$$x = \frac{kX - 2Y}{2X(X-1)} \qquad y = \frac{kX + 2Y}{2X(X-1)}.$$

$$Y^2 = X\left(X^2 + \left(\frac{k^2}{4} - 2\right)X + 1\right).$$

$P = \left(1, \frac{k}{2}\right)$, torsion point of order 4.

$$(x) \diamond (y) = 4(P) - 4(-P) = 8(P).$$

$$P \equiv -\frac{1}{4} \mod \mathbb{Z} + \tau\mathbb{Z} \qquad k \in \mathbb{R}$$

$$\tau = \mathrm{i}y_\tau \qquad k \in \mathbb{R}, |k| > 4,$$

$$\tau = \frac{1}{2} + \mathrm{i}y_\tau \qquad k \in \mathbb{R}, |k| < 4$$

Understand cycle $[|x| = 1] \in H_1(E, \mathbb{Z})$

$$\Omega = \tau\Omega_0 \quad k \in \mathbb{R}$$

$$-r(\{x,y\}) = -\int_\gamma \eta(x,y) = \mathrm{Im}\left(\frac{\Omega}{y_\tau \Omega_0} R_\tau\left((x)\diamond(y)\right)\right)$$

$$m(k) = \frac{4}{\pi}\,\mathrm{Im}\left(\frac{\tau}{y_\tau} R_\tau(-\mathrm{i})\right), \quad k \in \mathbb{R}$$

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Take $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SL_2(\mathbb{Z})$.

$$m(k) = -\frac{4|\tau|^2}{\pi y_\tau} J_{-\frac{1}{\tau}}\left(\mathrm{e}^{-\frac{2\pi \mathrm{i}}{4\tau}}\right)$$

If we let $\mu = -\frac{1}{4\tau}$, then

$$m(k) = -\frac{1}{\pi y_\mu} J_{4\mu}\left(\mathrm{e}^{2\pi\mathrm{i}\mu}\right)$$

$$= \mathrm{Re}\left(\frac{16 y_\mu}{\pi^2} \sum_{m,n}{}' \frac{\chi_{-4}(m)}{(m+n4\mu)^2(m+n4\bar{\mu})}\right)$$

Functional equations for the regulator

From

$$J(z) = p \sum_{x^p = z} J(x)$$

Let p prime,

$$(1 + \chi_{-4}(p)p^2) J_{4\tau}\left(\mathrm{e}^{2\pi\mathrm{i}\tau}\right) = \sum_{j=0}^{p-1} p J_{\frac{4(\tau+j)}{p}}\left(\mathrm{e}^{\frac{2\pi\mathrm{i}(\tau+j)}{p}}\right) + \chi_{-4}(p) J_{4p\tau}\left(\mathrm{e}^{2\pi\mathrm{i}p\tau}\right)$$

- In particular, $p = 2$,

$$J_{4\tau}\left(\mathrm{e}^{2\pi\mathrm{i}\tau}\right) = 2J_{2\tau}\left(\mathrm{e}^{\pi\mathrm{i}\tau}\right) + 2J_{2(\tau+1)}\left(\mathrm{e}^{\pi\mathrm{i}(\tau+1)}\right)$$

- Also:

$$J_{\frac{2\tau+1}{2}}\left(\mathrm{e}^{\pi\mathrm{i}\tau}\right) = J_{2\tau}\left(\mathrm{e}^{\pi\mathrm{i}\tau}\right) - J_{2\tau}\left(-\mathrm{e}^{\pi\mathrm{i}\tau}\right)$$

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Second degree modular equation, $|h| < 1$, $h \in \mathbb{R}$,

$$q^2\left(\left(\frac{2h}{1+h^2}\right)^2\right) = q\left(h^4\right).$$

$h \longrightarrow \mathrm{i}h$

$$-q\left(\left(\frac{2h}{1+h^2}\right)^2\right) = q\left(\left(\frac{2\mathrm{i}h}{1-h^2}\right)^2\right).$$

From

$$J_{4\tau}\left(\mathrm{e}^{2\pi\mathrm{i}\tau}\right) = 2J_{2\tau}\left(\mathrm{e}^{\pi\mathrm{i}\tau}\right) + 2J_{2(\tau+1)}\left(\mathrm{e}^{\pi\mathrm{i}(\tau+1)}\right)$$

One gets

$$m\left(q\left(\left(\frac{2h}{1+h^2}\right)^2\right)\right) + m\left(q\left(\left(\frac{2\mathrm{i}h}{1-h^2}\right)^2\right)\right) = m\left(q\left(h^4\right)\right).$$

$$m\left(2\left(h+\frac{1}{h}\right)\right) + m\left(2\left(\mathrm{i}h+\frac{1}{\mathrm{i}h}\right)\right) = m\left(\frac{4}{h^2}\right).$$

$$J_{\frac{2\mu+1}{2}}\left(\mathrm{e}^{\frac{2\pi\mathrm{i}\mu}{2}}\right) = J_{2\mu}\left(\mathrm{e}^{\pi\mathrm{i}\mu}\right) - J_{2\mu}\left(-\mathrm{e}^{\pi\mathrm{i}\mu}\right)$$

Set $\tau = -\frac{1}{2\mu}$ and use $\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$.

$$D_{\frac{\tau-1}{2}}(-\mathrm{i}) = D_\tau(-\mathrm{i}) - \frac{1}{y_{2(\mu+1)}} J_{2(\mu+1)}\left(\mathrm{e}^{\frac{2\pi\mathrm{i}(\mu+1)}{2}}\right)$$

First equation was:

$$D_{\frac{\tau-1}{2}}(-\mathrm{i}) = D_\tau(-\mathrm{i}) + \frac{1}{y_{2(\mu+1)}} J_{2(\mu+1)}\left(\mathrm{e}^{\frac{2\pi\mathrm{i}(\mu+1)}{2}}\right)$$

Putting things together,

$$2D_\tau(-\mathrm{i}) = D_{\frac{\tau}{2}}(-\mathrm{i}) + D_{\frac{\tau-1}{2}}(-\mathrm{i})$$

this is:

$$2m\left(2\left(h+\frac{1}{h}\right)\right) = m(4h^2) + m\left(\frac{4}{h^2}\right).$$

Hecke operators approach

$$m(k) = \mathrm{Re}\left(-\pi \mathrm{i}\mu + 2\sum_{n=1}^{\infty}\sum_{d|n}\chi_{-4}(d)d^2\frac{q^n}{n}\right)$$
$$= \mathrm{Re}\left(-\pi \mathrm{i}\mu - \pi\mathrm{i}\int_{\mathrm{i}\infty}^{\mu}(e(z)-1)\mathrm{d}z\right)$$

where

$$e(\mu) = 1 - 4\sum_{n=1}^{\infty}\sum_{d|n}\chi_{-4}(d)d^2q^n$$

is an Eisenstein series. Hence the equations can be also deduced from identities of Hecke operators.

Direct approach

Also some equations can be proved directly using isogenies:

$$\phi_1 : E_{2\left(h+\frac{1}{h}\right)} \to E_{4h^2}, \qquad \phi_2 : E_{2\left(h+\frac{1}{h}\right)} \to E_{\frac{4}{h^2}}.$$

$$\phi_1 : (X,Y) \to \left(\frac{X(h^2X+1)}{X+h^2}, -\frac{h^3Y\left(X^2+2h^2X+1\right)}{(X+h^2)^2} \right)$$

$$m\left(4h^2\right) = r_1\left(\{x_1, y_1\}\right) = \frac{1}{2\pi}\int_{|X_1|=1} \eta(x_1, y_1)$$

$$= \frac{1}{4\pi}\int_{|X|=1} \eta(x_1 \circ \phi_1, y_1 \circ \phi_1) = \frac{1}{2} r\left(\{x_1 \circ \phi_1, y_1 \circ \phi_1\}\right)$$

The identity with $h = \frac{1}{\sqrt{2}}$

$$m(2) + m(8) = 2m\left(3\sqrt{2}\right)$$

$$m\left(3\sqrt{2}\right) + m\left(\mathrm{i}\sqrt{2}\right) = m(8)$$

$f = \frac{\sqrt{2}Y - X}{2}$ in $\mathbb{C}(E_{3\sqrt{2}})$.

$$(f) \diamond (1-f) = 6(P) - 10(P+Q) \Rightarrow 6(P) \sim 10(P+Q).$$

$Q = \left(-\frac{1}{h^2}, 0\right)$ has order 2.

$$\phi : E_{3\sqrt{2}} \to E_{\mathrm{i}\sqrt{2}} \qquad (X, Y) \to (-X, \mathrm{i}Y)$$

$$r_{\mathrm{i}\sqrt{2}}(\{x, y\}) = r_{3\sqrt{2}}(\{x \circ \phi, y \circ \phi\})$$

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But

$$(x \circ \phi) \diamond (y \circ \phi) = 8(P + Q)$$

$$(x) \diamond (y) = 8(P)$$

$$6r_{3\sqrt{2}}(\{x, y\}) = 10r_{\mathrm{i}\sqrt{2}}(\{x, y\})$$

and

$$3m(3\sqrt{2}) = 5m(\mathrm{i}\sqrt{2}).$$

Consequently,

$$m(8) = \frac{8}{5}m(3\sqrt{2})$$

$$m(2) = \frac{2}{5}m(3\sqrt{2})$$

Other families

- Hesse family

$$h(a^3) = m\left(x^3 + y^3 + 1 - \frac{3xy}{a}\right)$$

(studied by Rodriguez-Villegas 1997)

$$h(u^3) = \sum_{j=0}^{2} h\left(1 - \left(\frac{1-\xi_3^j u}{1+2\xi_3^j u}\right)^3\right) \qquad |u| \text{ small}$$

- More complicated equations for examples studied by Stienstra 2005:

$$m\left((x+1)(y+1)(x+y) - \frac{xy}{t}\right)$$

and Bertin 2004, Zagier $<$ 2005, and Stienstra 2005:

$$m\left((x+y+1)(x+1)(y+1) - \frac{xy}{t}\right)$$