Mahler measure under variations of the base group

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Mahler measure for one-variable polynomials

Pierce (1918): $P \in \mathbb{Z}[x]$ monic,

$$P(x) = \prod_i (x - \alpha_i)$$

$$\Delta_n = \prod_i (\alpha_i^n - 1)$$

$$P(x) = x - 2 \Rightarrow \Delta_n = 2^n - 1$$

Lehmer (1933): Consider

$$\frac{\Delta_{n+1}}{\Delta_n}$$

$$\lim_{n\to\infty} \frac{|\alpha^{n+1}-1|}{|\alpha^n - 1|} = \begin{cases} |\alpha| & \text{if } |\alpha| > 1 \\ 1 & \text{if } |\alpha| < 1 \end{cases}$$

For

$$P(x) = a \prod_i (x - \alpha_i)$$

$$M(P) = |a| \prod_i \max\{1, |\alpha_i|\}$$

$$m(P) = \log M(P) = \log|a| + \sum_i \log^+ |\alpha_i|$$

Kronecker's Lemma

$P \in \mathbb{Z}[x]$, $P \neq 0$,

$$m(P) = 0 \Leftrightarrow P(x) = x^k \prod \Phi_{n_i}(x)$$

where Φ_{n_i} are cyclotomic polynomials

Lehmer's question

Lehmer (1933)

$$m(x^{10}+x^9-x^7-x^6-x^5-x^4-x^3+x+1)$$

$$=\log(1.176280818\dots)=0.162357612\dots$$

$$\sqrt{\Delta_{379}}=1,794,327,140,357$$

Does there exist $C>0$, for all $P(x)\in\mathbb{Z}[x]$

$$m(P)=0 \quad \text{or} \quad m(P)>C??$$

Is the above polynomial the best possible?

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Mahler measure of several variable polynomials

$P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, the (logarithmic) *Mahler measure* is :

$$\begin{aligned} m(P) &= \int_0^1 \dots \int_0^1 \log |P(\mathrm{e}^{2\pi \mathrm{i}\theta_1}, \dots, \mathrm{e}^{2\pi \mathrm{i}\theta_n})| \mathrm{d}\theta_1 \dots \mathrm{d}\theta_n \\ &= \frac{1}{(2\pi \mathrm{i})^n} \int_{\mathbb{T}^n} \log |P(x_1, \dots, x_n)| \frac{\mathrm{d}x_1}{x_1} \dots \frac{\mathrm{d}x_n}{x_n} \end{aligned}$$

Jensen's formula:

$$\int_0^1 \log |\mathrm{e}^{2\pi \mathrm{i}\theta} - \alpha| \mathrm{d}\theta = \log^+ |\alpha|$$

recovers one-variable case.

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Boyd & Lawton Theorem

$P \in \mathbb{C}[x_1, \dots, x_n]$

$$\lim_{k_2 \to \infty} \dots \lim_{k_n \to \infty} m(P(x, x^{k_2}, \dots, x^{k_n})) = m(P(x_1, x_2, \dots, x_n))$$

Examples in several variables

Smyth (1981)

- $$m(1+x+y) = \frac{3\sqrt{3}}{4\pi}\mathrm{L}(\chi_{-3},2) = \mathrm{L}'(\chi_{-3},-1) = \frac{D\left(\mathrm{e}^{\frac{\pi \mathrm{i}}{3}}\right)}{\pi}$$

- $$m(1+x+y+z) = \frac{7}{2\pi^2}\zeta(3)$$

$$\mathrm{L}(\chi_{-3},s) = \sum_{n=1}^{\infty}\frac{\chi_{-3}(n)}{n^s} \qquad \chi_{-3}(n) = \begin{cases} 1 & n \equiv 1 \bmod 3 \\ -1 & n \equiv -1 \bmod 3 \\ 0 & n \equiv 0 \bmod 3 \end{cases}$$

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$$

Boyd, Deninger, Rodriguez-Villegas (1997)

$$m\left(x+\frac{1}{x}+y+\frac{1}{y}-k\right)\stackrel{?}{=}\frac{\mathrm{L}'(E_k,0)}{B_k} \qquad k\in\mathbb{N}, \quad k\neq 4$$

E_k determined by $x+\frac{1}{x}+y+\frac{1}{y}-k=0$.

$$m\left(x+\frac{1}{x}+y+\frac{1}{y}-4\sqrt{2}\right)=\mathrm{L}'(E_{4\sqrt{2}},0)$$

$$E_{4\sqrt{2}}: Y^2 = X^3-44X+112$$

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The general technique

Rodriguez-Villegas (1997)

$$P_\lambda(x,y) = 1 - \lambda P(x,y) \qquad P(x,y) = x + \frac{1}{x} + y + \frac{1}{y}$$

$$m(P,\lambda) := m(P_\lambda)$$

$$m(P,\lambda) = \frac{1}{(2\pi \mathrm{i})^2}\int_{\mathbb{T}^2} \log|1-\lambda P(x,y)|\frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y}.$$

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Note

$$|\lambda P(x,y)| < 1, \qquad \lambda \quad \text{small}, \quad x, y \in \mathbb{T}^2$$

$$\tilde{m}(P,\lambda) = \frac{1}{(2\pi \mathrm{i})^2} \int_{\mathbb{T}^2} \log(1 - \lambda P(x,y)) \frac{\mathrm{d}x}{x} \frac{\mathrm{d}y}{y}$$

$$\frac{\mathrm{d}\tilde{m}(P,\lambda)}{\mathrm{d}\lambda} = -\frac{1}{(2\pi \mathrm{i})^2} \int_{\mathbb{T}^2} \frac{P(x,y)}{1 - \lambda P(x,y)} \frac{\mathrm{d}x}{x} \frac{\mathrm{d}y}{y}$$

Let

$$u(P,\lambda) = \frac{1}{(2\pi \mathrm{i})^2} \int_{\mathbb{T}^2} \frac{1}{1-\lambda P(x,y)} \frac{\mathrm{d}x}{x} \frac{\mathrm{d}y}{y}$$

$$= \sum_{n=0}^{\infty} \lambda^n \frac{1}{(2\pi \mathrm{i})^2} \int_{\mathbb{T}^2} P(x,y)^n \frac{\mathrm{d}x}{x} \frac{\mathrm{d}y}{y} = \sum_{n=0}^{\infty} a_n \lambda^n$$

Where

$$\frac{1}{(2\pi \mathrm{i})^2} \int_{\mathbb{T}^2} P(x,y)^n \frac{\mathrm{d}x}{x} \frac{\mathrm{d}y}{y} = [P(x,y)^n]_0 = a_n$$

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$$= -\int_0^\lambda (u(P,t)-1)\frac{\mathrm{d}t}{t} = -\sum_{n=1}^\infty \frac{a_n\lambda^n}{n}$$

In the case $P = x + \frac{1}{x} + y + \frac{1}{y}$,

$$a_n = 0 \qquad n \quad \text{odd}$$

$$a_{2m} = \binom{2m}{m}^2$$

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Definition

$\mathbb{F}_{x_1,\dots,x_l}$ *free group in* $x_1, \dots, x_l$,

$N \triangleleft \mathbb{F}_{x_1,\dots,x_l}$, $\Gamma = \mathbb{F}_{x_1,\dots,x_l}/N$

$$Q = Q(x_1, \dots, x_l) = \sum_{g \in \Gamma} c_g g \in \mathbb{C}\Gamma,$$

$$Q^* = \sum_{g \in \Gamma} \overline{c_g} g^{-1} \in \mathbb{C}\Gamma \textit{ reciprocal.}$$

$P = P(x_1, \dots, x_l) \in \mathbb{C}\Gamma$, $P = P^*$, $|\lambda| <$ *length of* P,

$$m_\Gamma(P, \lambda) = -\sum_{n=1}^{\infty} \frac{a_n \lambda^n}{n},$$

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We also write

$$u_\Gamma(P,\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n$$

for the generating function of the a_n.

$Q(x_1,\dots,x_l) \in \mathbb{C}\Gamma$

$$QQ^* = \frac{1}{\lambda}(1-(1-\lambda QQ^*))$$

for λ real and positive and $1/\lambda$ larger than the length of QQ^*.

$$m_\Gamma(Q) = -\frac{\log\lambda}{2} - \sum_{n=1}^{\infty}\frac{b_n}{2n}, \qquad b_n = [(1-\lambda QQ^*)^n]_0 \,.$$

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Lück's combinatorial L^2-torsion.

K knot

$$\Gamma = \pi_1(S^3 \setminus K) = \langle x_1, \ldots, x_g \mid r_1, \ldots, r_{g-1} \rangle$$

Let

$$F = \begin{pmatrix} \frac{\partial r_1}{\partial x_1} & \cdots & \frac{\partial r_1}{\partial x_g} \\ \vdots & \ddots & \vdots \\ \frac{\partial r_{g-1}}{\partial x_1} & \cdots & \frac{\partial r_{g-1}}{\partial x_g} \end{pmatrix} \in M^{(g-1)\times g}(\mathbb{C}\Gamma)$$

Fox matrix.

Delete a column $F \rightsquigarrow A \in M^{(g-1)\times(g-1)}(\mathbb{C}\Gamma)$.

Theorem

(Lück) Suppose K is a hyperbolic knot. Then, for k sufficiently large

$$\frac{1}{3\pi}\mathrm{Vol}(S^3 \setminus K) = 2(g-1)\ln(k) - \sum_{n=1}^{\infty}\frac{1}{n}\mathrm{tr}_{\mathbb{C}\Gamma}\left((1-k^{-2}AA^*)^n\right).$$

$A \in M^g\mathbb{C}[t, t^{-1}]$ the right-hand side is $2m(\det(A))$.

The Mahler measure over finite groups

$$P = \sum_i (\delta_i S_i + \overline{\delta_i} S_i^{-1}) + \sum_j \eta_j T_j \in \mathbb{C}\Gamma$$

$S_i \neq S_i^{-1}$, $T_j = T_j^{-1}$, $\delta_i \in \mathbb{C}$, $\eta_j \in \mathbb{R}$, and $S_i, T_j \in \Gamma$,

Theorem

For Γ finite

$$m_\Gamma(P, \lambda) = \frac{1}{|\Gamma|} \log \det(I - \lambda A),$$

A is the adjacency matrix of the Cayley graph (with weights) and $\frac{1}{\lambda} > \rho(A)$.

Analytic continuation for $m_\Gamma(P, \lambda)$ to $\mathbb{C} \setminus \mathrm{Spec}(A)$.

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Abelian Groups

Γ finite abelian group

$$\Gamma = \mathbb{Z}/m_1\mathbb{Z} \times \cdots \times \mathbb{Z}/m_l\mathbb{Z}$$

Corollary

$$m_\Gamma(P, \lambda) = \frac{1}{|\Gamma|} \log \left(\prod_{j_1, \dots, j_l} \left(1 - \lambda P(\xi_{m_1}^{j_1}, \dots, \xi_{m_l}^{j_l})\right) \right)$$

where ξ_k is a primitive root of unity.

Uses description of the spectra of Cayley graphs of finite groups given by Babai (1979)

Theorem

For small λ,

$$\lim_{m_1,\dots,m_l\to\infty} m_{\mathbb{Z}/m_1\mathbb{Z}\times\cdots\times\mathbb{Z}/m_l\mathbb{Z}}(P,\lambda) = m_{\mathbb{Z}^l}(P,\lambda).$$

Where the limit is with $m_1,\dots,m_l$ going to infinity independently.

Dihedral groups

$$\Gamma = D_m = \langle \rho, \sigma \mid \rho^m, \sigma^2, \sigma\rho\sigma\rho \rangle.$$

Theorem

Let $P \in \mathbb{C}[D_m]$ be reciprocal. Then

$$[P^n]_0 = \frac{1}{2m} \sum_{j=1}^{m} \left(P^n\left(\xi_m^j, 1\right) + P^n\left(\xi_m^j, -1\right)\right),$$

where P^n is expressed as a sum of monomials ρ^k, $\sigma\rho^k$ before being evaluated.

For $\Gamma = \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} = \langle x, y \,|\, x^m, y^2, [x, y] \rangle$,

$$[P^n]_0 = \frac{1}{2m} \sum_{j=1}^{m} \left(P\left(\xi_m^j, 1\right)^n + P\left(\xi_m^j, -1\right)^n \right).$$

Compare D_m and $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ with $x = \rho$ and $y = \sigma$ in D_m.

Theorem

Let

$$P = \sum_{k=0}^{m-1} \alpha_k x^k + \sum_{k=0}^{m-1} \beta_k y x^k$$

with real coefficients and reciprocal in $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ (therefore it is also reciprocal in D_m). Then

$$m_{\mathbb{Z}/m\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}}(P, \lambda) = m_{D_m}(P, \lambda).$$

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$$m_{\mathbb{Z}/m\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}}(P, \lambda) = m_{D_m}(P, \lambda).$$

Corollary

Let $P \in \mathbb{R}[\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}]$ *be reciprocal. Then*

$$m_{\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}}(P,\lambda) = m_{D_\infty}(P,\lambda),$$

where $D_\infty = \langle \rho, \sigma \mid \sigma^2, \sigma\rho\sigma\rho \rangle$.

Quotient approximations of the Mahler measure

Γ_m are quotients of Γ:

Theorem

Let $P \in \Gamma$ reciprocal.

- *For $\Gamma = D_\infty$, $\Gamma_m = D_m$,*

$$\lim_{m \to \infty} m_{D_m}(P, \lambda) = m_{D_\infty}(P, \lambda).$$

- *For $\Gamma = PSL_2(\mathbb{Z}) = \langle x, y \,|\, x^2, y^3 \rangle$, $\Gamma_m = \langle x, y \,|\, x^2, y^3, (xy)^m \rangle$,*

$$\lim_{m \to \infty} m_{\Gamma_m}(P, \lambda) = m_{PSL_2(\mathbb{Z})}(P, \lambda).$$

- *For $\Gamma = \mathbb{Z} * \mathbb{Z} = \langle x, y \rangle$, $\Gamma_m = \langle x, y \,|\, [x, y]^m \rangle$,*

$$\lim_{m \to \infty} m_{\Gamma_m}(P, \lambda) = m_{\mathbb{Z} * \mathbb{Z}}(P, \lambda).$$

Arbitrary number of variables

For $P_{1,l} = x_1 + x_1^{-1} + \cdots + x_l + x_l^{-1}$,

$$u_{\mathbb{F}_l}(P_{1,l}, \lambda) = g_{2l}(\lambda).$$

where

$$g_d(\lambda) = \frac{2(d-1)}{d - 2 + d\sqrt{1 - 4(d-1)\lambda^2}}.$$

is the generating function of the circuits of a d-regular tree (Bartholdi, 1999).

For $P_{2,l} = (1 + x_1 + \cdots + x_{l-1})\left(1 + x_1^{-1} + \cdots + x_{l-1}^{-1}\right)$,

$$u_{\mathbb{F}_{l-1}}(P_{2,l}, \lambda) = g_l(\lambda).$$

In particular,

$$m_{\mathbb{F}_l}(P_{1,l}, \lambda) = m_{\mathbb{F}_{2l-1}}(P_{2,2l}, \lambda).$$

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Abelian case.

For $P_{1,l} = x_1 + x_1^{-1} + \cdots + x_l + x_l^{-1}$,

$$\left[P_{1,l}^n\right]_0 = \sum_{a_1+\cdots+a_l=n} \frac{(2n)!}{(a_1!)^2 \ldots (a_l!)^2},$$

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$$\left[P_{1,l}^{2n}\right]_0 = \binom{2n}{n} \left[P_{2,l}^n\right]_0$$

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Further Study

- Why are m_Γ, u_Γ "too nice", i.e., algebraic, or coefficients satisfy recurrences?
- Is the new Mahler measure multiplicative?
- Further studies with variations of the base group.
- What can we say of the combinatorial L^2-torsion?

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