On a conjecture by Boyd

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Canadian Number Theory Association X Meeting
University of Waterloo
July 14th, 2008

Mahler measure of several variable polynomials

$P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, the (logarithmic) *Mahler measure* is :

$$\begin{aligned} m(P) &= \int_0^1 \dots \int_0^1 \log |P(e^{2\pi i\theta_1}, \dots, e^{2\pi i\theta_n})| d\theta_1 \dots d\theta_n \\ &= \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log |P(x_1, \dots, x_n)| \frac{dx_1}{x_1} \dots \frac{dx_n}{x_n}. \end{aligned}$$

By Jensen's formula,

$$m\left(a \prod (x - \alpha_i)\right) = \log |a| + \sum \log \max\{1, |\alpha_i|\}.$$

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Examples in several variables

Smyth (1981)

- $$m(1+x+y) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2) = L'(\chi_{-3}, -1)$$
- $$m(1+x+y+z) = \frac{7}{2\pi^2}\zeta(3)$$

The measures of a family of genus-one curves

$$m(k) := m\left(x + \frac{1}{x} + y + \frac{1}{y} + k\right)$$

Boyd (1998)

$$m(k) \stackrel{?}{=} s_k L'(E_k, 0) \qquad k \in \mathbb{N} \neq 0, 4 \quad s_k \in \mathbb{Q}^\times$$

E_k elliptic curve, projective closure of

$$x + \frac{1}{x} + y + \frac{1}{y} + k = 0.$$

elliptic modular surface associated to $\Gamma_0(4)$.

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Deninger (1997)

L-functions $\leftarrow$ Bloch-Beilinson's conjectures

Rodriguez-Villegas (1997)

$k = 4\sqrt{2}$

$$m\left(4\sqrt{2}\right) = m\left(x + \frac{1}{x} + y + \frac{1}{y} + 4\sqrt{2}\right) = L'(E_{4\sqrt{2}}, 0)$$

$k = 3\sqrt{2}$

$$m\left(3\sqrt{2}\right) = m\left(x + \frac{1}{x} + y + \frac{1}{y} + 3\sqrt{2}\right) = \frac{5}{2}L'(E_{3\sqrt{2}}, 0)$$

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Kurokawa & Ochiai (2005)
For $h \in \mathbb{R}^*$,

$$m(4h^2) + m\left(\frac{4}{h^2}\right) = 2m\left(2\left(h + \frac{1}{h}\right)\right).$$

L. & Rogers (2007)
For $|h| < 1$, $h \neq 0$,

$$m\left(2\left(h + \frac{1}{h}\right)\right) + m\left(2\left(\mathrm{i}h + \frac{1}{\mathrm{i}h}\right)\right) = m\left(\frac{4}{h^2}\right).$$

Identities conjectured by Boyd

- L. & Rogers (2007)

$$m(8) = 4m(2)\left(= \frac{8}{5}m(3\sqrt{2}) = 4L'(E_{3\sqrt{2}}, 0)\right)$$

- L. (2008)

$$m(5) = 6m(1)$$

Regulators and Mahler measures

$$m(k) = -\frac{1}{2\pi} \int_{\mathbb{T}^1=\{|x|=1\}\subset\{P=0\}} \eta(x,y),$$

$$\eta(x,y) := \log|x| \mathrm{di} \arg y - \log|y| \mathrm{di} \arg x$$

Regulator map (Beilinson, Bloch):

$$r : K_2(E) \otimes \mathbb{Q} \to H^1(E, \mathbb{R})$$

$$\{x, y\} \to \left\{\gamma \to \int_\gamma \eta(x, y)\right\}$$

for $\gamma \in H_1(E, \mathbb{Z})$.
Matsumoto Theorem:

$$K_2(F) = \left\langle \{a, b\}, a, b \in F^\times \right\rangle / \langle \text{bilinear}, \{a, 1 - a\}\rangle$$

(F field)
Need integrality conditions, trivial tame symbols...

Computing the regulator

$$-\int_{\gamma} \eta(x, y) = D_E\left((x) \diamond (y)\right)$$

D_E is the elliptic dilogarithm.

$$(x) = \sum m_i(a_i), \qquad (y) = \sum n_j(b_j).$$

$$\mathbb{C}(E)^* \otimes \mathbb{C}(E)^* \to \mathbb{Z}[E(\mathbb{C})]^-$$

$$(x) \diamond (y) = \sum m_i n_j (a_i - b_j).$$

$$\mathbb{Z}[E(\mathbb{C})]^- = \mathbb{Z}[E(\mathbb{C})]/\sim \quad [-P] \sim -[P].$$

$$E_k : Y^2 = X\left(X^2 + \left(\frac{k^2}{4} - 2\right)X + 1\right),$$

$$x = \frac{kX - 2Y}{2X(X-1)}, \qquad y = \frac{kX + 2Y}{2X(X-1)}.$$

$E_k(\mathbb{Q}(k))_{\mathrm{tor}} \cong \mathbb{Z}/4\mathbb{Z}$. $P = \left(1, \frac{k}{2}\right)$.

$$(x) \diamond (y) = 8(P).$$

Proof of the functional equations

$$m(4h^2) + m\left(\frac{4}{h^2}\right) = 2m\left(2\left(h + \frac{1}{h}\right)\right).$$

$$\phi_1 : E_{2\left(h+\frac{1}{h}\right)} \to E_{4h^2}, \qquad \phi_2 : E_{2\left(h+\frac{1}{h}\right)} \to E_{\frac{4}{h^2}}.$$

$$\phi_1 : (X, Y) \to \left(\frac{X(h^2X+1)}{X+h^2}, -\frac{h^3Y\left(X^2+2h^2X+1\right)}{(X+h^2)^2}\right)$$

$$m\left(4h^2\right) = r_1\left(\{x_1, y_1\}\right) = \frac{1}{2}r\left(\{x_1 \circ \phi_1, y_1 \circ \phi_1\}\right)$$

$$(x_1 \circ \phi_1) \diamond (y_1 \circ \phi_1) = 16(P) - 16(P+Q),$$

$$(x_2 \circ \phi_2) \diamond (y_2 \circ \phi_2) = 16(P) + 16(P+Q).$$

Where $Q = \left(-\frac{1}{h^2}, 0\right)$ is a point of order 2.

$$\frac{1}{2} r\left(\{x_1 \circ \phi_1, y_1 \circ \phi_1\}\right) + \frac{1}{2} r\left(\{x_2 \circ \phi_2, y_2 \circ \phi_2\}\right) = 2r\left(\{x_0, y_0\}\right),$$

and therefore

$$r_1\left(\{x_1, y_1\}\right) + r_2\left(\{x_2, y_2\}\right) = 2r\left(\{x_0, y_0\}\right).$$

$$m\left(2\left(h+\frac{1}{h}\right)\right)+m\left(2\left(\mathrm{i}h+\frac{1}{\mathrm{i}h}\right)\right)=m\left(\frac{4}{h^2}\right).$$

$$\phi : E_{2\left(h+\frac{1}{h}\right)} \to E_{2\left(\mathrm{i}h+\frac{1}{\mathrm{i}h}\right)}, \qquad (X,Y)\to(-X,\mathrm{i}Y),$$

$$r_{2\left(\mathrm{i}h+\frac{1}{\mathrm{i}h}\right)}(\{x,y\}) = r_{2\left(h+\frac{1}{h}\right)}(\{x\circ\phi, y\circ\phi\}).$$

$$(x\circ\phi)\diamond(y\circ\phi)=8(P+Q).$$

Proof of the identities

$h = \frac{1}{\sqrt{2}}$ in both equations,

$$m(2) + m(8) = 2m\left(3\sqrt{2}\right).$$

$$m\left(3\sqrt{2}\right) + m\left(\mathrm{i}\sqrt{2}\right) = m(8).$$

$h = \frac{1}{2}$:

$$m(1) + m(16) = 2m(5).$$

$$m(5) + m(-3\mathrm{i}) = m(16).$$

Need additional relations!

(Between (P) and $(P+Q)$)

$$f = \frac{Y}{2h} + \left(\frac{1}{2} - \frac{1}{2h^2}\right) X.$$

$$(f) = (2P) + 2(P+Q) - 3O.$$

$$(1-f) = (P) + (A) + (B) - 3O,$$

$$A = \left(\frac{-3+\sqrt{9-16h^2}}{2}, \frac{7h}{2} - \frac{3}{2h} - \left(h - \frac{1}{h}\right)\frac{\sqrt{9-16h^2}}{2}\right),$$

$$B = A^{\sigma}$$

$h = \frac{1}{\sqrt{2}}$

$$A = 3P + Q, \qquad B = Q,$$

$$(f) \diamond (1 - f) = 6(P) - 10(P + Q) \sim 0$$

$h = \frac{1}{2}$,

$$2A = 2B = P, \quad B - A = 2P, \quad A + B = -P.$$

$$(f) \diamond (1 - f) = 2(Q + A) + 2(Q + B) - 6(P + Q) + 4(P) + 2(A) + 2(B).$$

$$g = \frac{\sqrt{5}-1}{10}Y + \frac{3+\sqrt{5}}{20}(X+4)$$

$$(g) = (Q) + (A) + (-Q-A) - 3O$$

$$(1-g) = (-P) + 2(B) - 3O$$

$$(g)\diamond(1-g) = 3(Q+P) - 2(Q+B) - 3(A) + 4(Q+A) - 3(P) + 5(B).$$

$$(g^\sigma)\diamond(1-g^\sigma) = 3(Q+P) - 2(Q+A) - 3(B) + 4(Q+B) - 3(P) + 5(A).$$

$$(f)\diamond(1-f) - (g)\diamond(1-g) - (g^\sigma)\diamond(1-g^\sigma) = -12(Q+P) + 10(P) \sim 0$$

Finally,

$$m\left(x+\frac{1}{x}+y+\frac{1}{y}+8\right)=4m\left(x+\frac{1}{x}+y+\frac{1}{y}+2\right)$$

$$m\left(x+\frac{1}{x}+y+\frac{1}{y}+5\right)=6m\left(x+\frac{1}{x}+y+\frac{1}{y}+1\right)$$