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Notre but est d'essayer faire un grand partie de mon livre — ici sont les sujets. J'espere nous pouvons couvrir vers 75%

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